Asset Bubbles and Inflation as Competing Monetary Phenomena

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Abstract. In an economy in which adjusting prices comes at a fixed menu cost, a standard Taylor rule generates multiple equilibria with varying price rigidity, inflation, and real interest rate, including equilibria in which bubbles arise even though the interest rate that would prevail in the absence of nominal rigidities would not be sufficiently low to sustain them. In our setup, these policy-induced bubbles differ from natural ones in three important ways: i) They earn low returns; ii) They are incompatible with high CPI inflation, and so they burst when inflation picks up; iii) Once issued, they always crowd out investment by draining resources from the most financially constrained agents.

Introduction

An increasing number of observers contend that the very accommodative monetary policies that have prevailed in advanced economies since 2008 have had the unintended consequences of blowing asset bubbles instead of spurring much needed real investment. This narrative has gained significant traction since the 2010 round of asset purchases by the Federal Reserve dubbed QE2, and even more so since the Covid-19 crisis.[1] The proponents of this view accordingly worry that these bubbles may burst now that inflation has picked up, thereby generating severe financial instability.[2]

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[1] In 2010 for example, four top Republican congressmen wrote to Chairman Bernanke that QE2 could "potentially generate artificial asset bubbles that could cause further economic disruptions". Similar reactions abound in the blogosphere since the pandemic: "The markets are alive with the sound of echo bubbles" (https://on.ft.com), "Strict Inflation Targets For Central Banks Have Caused Economic Harm" (https://on.ft.com) "Pandemic-Era Central Banking Is Creating Bubbles Everywhere" (https://www.bloomberg.com), "The Fed Has Created A Monster Bubble It Can No Longer Control" (https://seekingalpha.com), "The Fed Is Creating A Monster Bubble" (https://www.forbes.com/), "Fed Trying To Inflate A 4th Bubble To Fix The Third" (https://seekingalpha.com),"The Fed's Corporate Bond Buying Is Stoking Bubble Fears" (https://www.cnbc.com).

[2] See, e.g., "Stock Market Bubble Will Burst And Inflation Will Follow", "Will Higher Inflation End U.S. Asset Bubbles?" (https://www.forbes.com), "The End Of The "Everything Bubble" Could Destroy $75trn Of Assets" (https://moneyweek.com).

This narrative is commonly dismissed as not grounded in economic theory. In order to be sustainable, bubbles must earn an expected return that is below the rate at which the economy grows, and controlling real asset returns for an indefinite time span is viewed as beyond the reach of monetary policy. Monetary policy therefore cannot be the main cause of asset bubbles.

To be sure, a sizeable literature studies the interplay of monetary policy and bubbles.[3] Bubbles in this literature are not a monetary phenomenon, however. They arise because the natural interest rate—the one that would prevail in the presence of flexible prices—is sufficiently low to make bubbles sustainable. This literature then studies if and how monetary policy should be amended to take these "natural" bubbles into account.

By contrast, this paper introduces bubbles as *a pure consequence of monetary policy*. Such policy-induced bubbles may rise even when natural bubbles would be impossible in the absence of any nominal rigidity. Beyond fitting a widespread narrative, policy-induced bubbles have three important features that distinguish them from natural ones:

1. *Policy-induced bubbles do not lift interest rates nor asset returns.* Natural bubbles push the interest rate up, the more so the larger they are. By contrast, the interest rate remains low in the presence of policy-induced bubbles no matter their size. In particular, these bubbles earn themselves a low expected return.

2. *Policy-induced bubbles and CPI inflation are mutually exclusive.* Natural bubbles push CPI inflation up in our setup. By contrast, policy-induced bubbles and high CPI inflation cannot jointly occur in equilibrium. As an example, we construct a sunspot equilibrium in which policy-induced bubbles burst when inflation (stochastically) picks up.

3. *Policy-induced bubbles crowd investment out once issued.* In our particular setup, natural bubbles may (or may not) durably crowd investment in by alleviating financial constraints, whereas policy-induced bubbles are always substitutes to investment once issued.

Given these properties, policy-induced bubbles offer a natural rationalization of the common narrative according to which low policy rates may backfire into "bad" bubbles. The low return on these bubbles and their negative impact on investment, which are two sides of the same coin, fit well in an environment in which business investment has remained subdued despite low rates and compressed risk and liquidity premia. The incompatibility of policy-induced bubbles with sizeable CPI inflation also resonates with current concerns about market crashes following the rise of inflation.

[3] Following Bernanke and Gertler (2001), contributions include Galí (2014), Ikeda and Phan (2016), Dong et al. (2020), Asriyan et al. (2021), and Allen et al. (2022).

Interestingly, our theory of bubbles as a monetary phenomenon relies solely on the interaction of two very standard textbook monetary ingredients. First, price setters incur a fixed menu cost when adjusting prices. Second, monetary policy consists in a basic Taylor rule that makes the policy rate contingent on realized inflation. The reason these two ingredients unlock bubbles as a monetary phenomenon is as follows. The starting point is the insight in Ball and Romer (1991) that if prices are strategic complements, fixed menu costs may open up the possibility of multiple equilibria with varying price rigidity because each firm benefits more from adjusting its prices if other firms do so. The number of such equilibria as well as their respective real and nominal characteristics depend on the conduct of monetary policy, and this creates a role for the Taylor rule. We show that a basic Taylor rule implies that our economy admits multiple equilibria across which firms adjust prices at different frequencies. Equilibria with more rigid prices also feature a lower inflation and a lower real interest rate. The interest rate in the most rigid equilibria may be so low that bubbles arise in them even though they would not be possible in the more flexible equilibria.

More formally, suppose that a monetary authority commits to a Taylor rule with parameters $r^M, \Pi^M, \psi > 0$. That is, it commits to a nominal rate between t and $t+1$:

$$R_t = r^M \Pi^M \left(\frac{\Pi_t}{\Pi^M} \right)^{1+\psi},$$

where Π_t is the realization of inflation at date t. Then any steady-state equilibrium real rate r and inflation Π must be such that the Fisher equation and this Taylor rule give consistent values of the nominal rate: $r\Pi = R_t$, or

$$\Pi \left(r^M \right)^{\frac{1}{\psi}} = \Pi^M r^{\frac{1}{\psi}}$$

Workhorse New Keynesian models with time-dependent price rigidity typically admit a unique such steady state (r, Π). If nominal rigidity consists in a fixed menu cost, there are by contrast several such equilibrium pairs (r, Π) that satisfy the above relation together with all the individual rationality and market-clearing conditions characterizing equilibrium. The value of the real rate, that of inflation, and the frequency of price adjustment comove across equilibria. In the fixed-price equilibrium in particular, which can be (stochastically) temporary, bubbles may rise that do not move the equilibrium interest rate.

It is important to stress that in our theory, *bubbles as a pure monetary phenomenon do not arise because the monetary authority permanently controls real interest rates.* They do so for the opposite reason that a Taylor rule generates indeterminacy. Monetary policy only imposes that the real rate and inflation be low whenever the private sector coordinates on infrequent price adjustment. The monetary authority however has no control over the degree of price rigidity on which firms coordinate. In particular, this degree can vary over time in a stochastic fashion, as is the case in some of the equilibria that we construct.

At the effective lower bound. We also show that when monetary policy is constrained by an effective lower bound, the causality between bubbles and price rigidity may interestingly become two-sided. Not only is price rigidity necessary to ensure that policy-induced bubbles that do not affect the real rate can arise, as is generically the case. It may also be that equilibria with rigid prices can only arise in the presence of these policy-induced bubbles. This double feedback between policy-induced bubbles and inflation expectations is due to the fact that bubbles, by crowding out investment and shrinking entrepreneurs' profits, make price adjustment less profitable to them, thereby making the rigid-price equilibrium sustainable.

The paper is organized as follows. Section 1 sets up the model. Sections 2 studies its flexible-price equilibria, describing in particular the bubbles deemed "natural" that can arise when prices are flexible. Section 3 studies equilibria with more rigid prices and discusses equilibrium multiplicity. It dwells on the bubbles deemed "policy-induced" that may arise when prices are rigid. Section 4 applies the results in Section 3 to the study of two particular economies that display interesting feedbacks between CPI inflation and asset bubbles. Section 5 concludes.

Related Literature

A very large and growing literature explores the empirical plausibility of menu costs as a significant source of price rigidity. Reviewing it is beyond the scope of this paper. Yet, the insight pioneered by Ball and Romer (1991) that menu costs may generate multiple equilibria has been much less explored. (Exceptions include John and Wolman (2008).) This paper is to my knowledge the first to stress that such multiplicity may go beyond inflation dynamics and pave the way to asset bubbles as a monetary phenomenon.

It is important to highlight that this fixed menu cost is the only source of equilibrium multiplicity that we focus on. In particular, the resulting multiplicity in inflation dynamics is unrelated to that possibly generated by interest-rate feedback rules in the presence of a lower bound (Benhabib et al., 2001a,b, 2002a,b). It could potentially prevail under any other modeling of monetary policy.[4] The Taylor rule here only delivers that the real rate is lower in equilibria with more (endogenous) nominal rigidities. This unlocks the possibility of policy-induced bubbles even when the natural rate is large.

This paper also has points of contact with two contributions in monetary economics. Galí-(2014) studies how a Taylor rule should be amended in the presence of a bubble. By contrast, I take a basic Taylor rule as given and focus on its implications for equilibrium multiplicity. Also, equilibrium multiplicity is only due to the possibility of bubbles in Galí (2014)) in which the

[4]Ball and Romer (1991) obtain this multiplicity in their original paper in which monetary policy consists in controlling money supply in the presence of a cash-in-advance constraint.

nominal rigidity is that prices must be set in advance. By contrast, a fixed menu cost generates multiplicity here even when bubbles cannot be sustained. In a recent contribution, Beaudry et al. (2023) study an economy that has several steady states in its real version, and show how monetary policy can have a long-lasting impact on the real rate in this case. By contrast, the version of our model without nominal rigidities admits a unique steady state in the absence of bubbles.

This paper also contributes to the literature that studies the effect of bubbles on investment, in particular in the presence of financial constraints (Farhi and Tirole, 2012a; Martin and Ventura, 2012; Aoki et al., 2014; Hirano and Yanagawa, 2016). We introduce simple monetary ingredients in a related environment that enables us to compare how natural and policy-induced bubbles affect investment.

This paper also has connections to the literature that studies interest-rate policies as a tool to mitigate financial-market imperfections (Benmelech and Bergman, 2012; Caballero and Simsek, 2020; Diamond and Rajan, 2012; Farhi and Tirole, 2012b). This literature has emphasized how subsidizing the interest rate and financial repression may backfire into various forms of excessive risk taking. To our knowledge, we are the first to show that such excessive risk taking may materialize into "bad" rational bubbles.

Finally, it is interesting to relate this paper to the intermediary asset pricing literature pioneered by He and Krishnamurthy (2012, 2013). In this literature, negative shocks to sophisticated investors' wealth negatively affect all asset prices. The very distinct impacts of natural and policy-induced bubbles on entrepreneurs' net worth is also the main driver of their respective properties here.

1 Setup

Our model is an elementary monetary version of an overlapping-generations economy in which the limited pledgeability of future cash flows may lead to the emergence of bubbles despite dynamic efficiency, as in Farhi and Tirole (2012a) or Martin and Ventura (2012). We focus on friction-driven bubbles in a dynamically efficient economy (as opposed to bubbles simply stemming from dynamic inefficiency) because i) unlike interest rates, the return on private capital has been seemingly larger than the growth rate of output in the US over the last two decades (Reis, 2021); ii) it is well-known that natural bubbles in this case can be complement to investment, which makes them an interesting benchmark for policy-induced ones.

Time is discrete and indexed by $t \in \mathbb{N}$. The economy is populated by private agents—households and entrepreneurs, and by a monetary authority. All agents use the same currency as a unit of account only ("cashless economy"). Private agents consume a final good that they produce out of a continuum of intermediate goods indexed by $i \in [0,1]$ using the technology

$$C_t = \left(\int_0^1 C_{i,t}^{1-\frac{1}{\epsilon}} di \right)^{\frac{\epsilon}{\epsilon-1}}, \tag{1}$$

where $\epsilon \geq 2$. The date-t price of intermediate good i is denoted P_t^i, and P_t denotes the price of the final good—the "price level".

Entrepreneurs. At each date, a unit mass of entrepreneurs are born and live for two dates. They consume only when old, at which time they are risk neutral. Entrepreneurs are endowed with a production technology and with an investment technology.

Production technology. Each date-t young entrepreneur $i \in [0,1]$ owns a technology that transforms L units of date-t labor into αL units of the date-t intermediate good i, where $\alpha > 0$. The technology fully depreciates after one production cycle.

Investment technology. Each date-t young entrepreneur owns a technology that transforms x date-t consumption units into ρx date-$(t+1)$ consumption units, where $\rho > 1$. That this return on investment ρ is strictly larger than the unit growth rate of the economy implies dynamic efficiency.

Households. A unit mass of households are born at each date, and live for two dates. Households supply labor to firms when young, and receive a large (exogenous) endowment when old.[5] They rank bundles (C^Y, C^O, L) of consumption when young, consumption when old, and labor according to the criterion

$$u(C^Y) + \beta C^O - \frac{\gamma L^2}{2}, \tag{2}$$

where $\beta \in (0,1)$, $\gamma > 0$, and u' exists and is a decreasing bijection over $(0, +\infty)$.

We will stick to the assumption in the baseline New Keynesian model of a perfect labor market cleared by a flexible wage (Galí, 2008; Woodford, 2003, e.g.). As in these models, this enables us to study equilibrium prices and quantities of goods in the simplest framework. On the other hand, we do not claim any serious analysis of labor markets.

Frictions. In addition to imperfect competition in the markets for intermediate goods, the economy is plagued by two frictions, a financial one and a monetary one.

Assumption 1. ***(Financial friction: Limited pledgeability)*** *An entrepreneur can divert all or part of the proceeds from her investment and consume a fraction $1-\lambda$ of the diverted proceeds, where $\lambda \in (0,1)$.*

[5] This endowment merely simplifies the exposition by ensuring that a positive-consumption constraint never binds. It will play no other role.

This financial friction will induce credit rationing that may give rise to bubbles under some circumstances despite dynamic efficiency ($\rho > 1$). The second friction is a nominal rigidity such that monetary policy may have real effects:

Assumption 2. ***(Monetary friction: Menu cost)*** *Young date-t entrepreneur $i \in [0,1]$ must incur a fixed cost equal to f consumption units, where $f \geq 0$, in order to change the price of intermediate good i from the statu quo P_{t-1}^i to a new value.*

The usual broad interpretation of the menu cost f is that it stands for the costs of information collection and decision making incurred by an entrepreneur unwilling to stick to the statu quo.[6] We posit for simplicity that for all $i \in [0,1]$, $P_{-1}^i = P_{-1} > 0$ exogenously given.

Monetary authority. At the outset, the monetary authority commits to a standard interest-rate feedback rule making nominal rates contingent on realized inflation. The rule consists in a gross nominal interest rate R_t on one-period nominal bonds between t and $t+1$ equal to

$$R_t = r^M \Pi^M \left(\frac{\Pi_t}{\Pi^M} \right)^{1+\psi}, \tag{3}$$

where $r^M,\ \Pi^M,\ \psi > 0$, and $\Pi_t = P_t / P_{t-1}$ is the (gross) rate of inflation between $t-1$ and t. The monetary authority has an aggregate net supply of one-period nominal bonds equal to zero at each date. We abstract from any lower-bound constraint on the policy rate throughout the analysis except in Section 4.2. How the parameters r^M and Π^M may be determined and relate to the economy will be discussed in due course.

Finally, we impose for brevity the parameter restrictions

$$u'\left(\frac{\alpha^2 \beta \rho}{\gamma} \right) < \beta \rho, \tag{4}$$

$$f \leq \frac{\lambda(\epsilon - 1)}{\epsilon^2} \frac{\alpha^2 \beta \rho}{\gamma}, \tag{5}$$

and will also explain their roles in due course.

Discussion. Two comments are in order. First, overlapping generations are only a simple way to generate the incompleteness that opens up the possibility of bubbles. Less stylized (and less tractable) alternatives would of course be available (Aiyagari, 1994; Bewley, 1986; Woodford, 1990, e.g.). As in Martin and Ventura (2012) or Farhi and Tirole (2012a), the concept of generation in this setup should not be interpreted literally: Time elapsing between two dates is much shorter than 75 years. Assuming the same short-lived agents with simple preferences as in these

[6] See Alvarez et al. (2011) for an explicit modelling of costly information collection in a price-setting problem.

papers enables us to introduce in the simplest fashion our novel insights on the joint instability of goods and assets markets. A particular gain from such simple preferences is that they enable us to characterize equilibria without resorting to log-linearization nor any other approximation, which seems desirable for a theoretical contribution.

Second, as in the seminal paper of Ball and Romer (1991), we rely on a simple fixed menu cost to generate multiple equilibria with varying price rigidity. We could alternatively borrow from the literature that generates such multiplicity out of informational frictions (Amador and Weill, 2010; Gaballo, 2017, e.g.). We leave this exciting route for future research.

2 Flexible-price equilibria

This section studies the existence and the properties of flexible-price equilibria in which entrepreneurs pay the cost f to adjust their prices at each date. Section 2.1 first shows that there exists at most one non-bubbly flexible-price equilibrium. Section 2.2 then discusses the existence and properties of the bubbles that may arise in the presence of flexible prices, deemed "natural" bubbles.

2.1 Non-bubbly equilibrium

I define a perfect-foresight equilibrium in a standard way as a situation in which private agents optimize with perfect foresight, markets clear, the monetary authority enforces the Taylor rule and has a zero-net supply of bonds, and $\log \Pi_t$ is bounded.[7]

This section shows that there exists at most one such equilibrium with flexible prices and without bubbles in two steps. It first assumes that entrepreneurs find it optimal to adjust their prices at each date and solves for the resulting equilibrium. It then checks that they find it indeed optimal to do so in equilibrium. The full-fledged equilibrium derivation is in Appendix A.1. The main steps are summarized below. Appendix A.1 shows that the real block of the model is time-invariant and so we drop the time subscripts for real variables for notational simplicity.[8]

[7]This latter restriction to non-exploding inflation is only meant to address the well-known criticism of the elusive terminal condition that applies to any model of inflation determination with a Taylor rule (Castillo-Martinez and Reis, 2019, e.g.).

[8]This stems essentially from absence of capital accumulation and quasi-linear preferences shutting down any connection between dates other than through expectations.

Households' supply of labor and savings. Denoting W_t the nominal wage, each date-t household selects a nominal investment in bonds B_t and a labor supply L that solve:

$$\max_{B_t, L} u(C^Y) + \beta C^O - \frac{\gamma L^2}{2} \tag{6}$$

s.t.

$$P_t C^Y + B_t \leq W_t L, \tag{7}$$

$$P_{t+1} C^O \leq R_t B_t + P_{t+1} e, \tag{8}$$

$$C^Y, C^O, L \geq 0, \tag{9}$$

where e is the household's exogenous endowment when old. Optimal labor supply yields

$$W_t u'(C^Y) = P_t \gamma L_t, \tag{10}$$

and optimal bond investment yields

$$P_{t+1} u'(C^Y) = \beta R_t P_t. \tag{11}$$

Thus the real rate r satisfies $r = R_t P_t / P_{t+1} = u'(C^Y)/\beta$.

Inflation. The Fisher equation (11) combined with the Taylor rule (3) yields for all $t \geq 0$

$$r \Pi_{t+1} = R_t = r^M \Pi^M \left(\frac{\Pi_t}{\Pi^M} \right)^{1+\psi}, \tag{12}$$

or

$$\frac{\Pi_{t+1}}{\Pi_t} = \frac{r^M}{r} \left(\frac{\Pi_t}{\Pi^M} \right)^{\psi}. \tag{13}$$

The only price path that satisfies this and does not lead to exploding inflation rates is such that

$$\Pi_t = \Pi^* \equiv \Pi^M \left(\frac{r}{r^M} \right)^{\frac{1}{\psi}} \tag{14}$$

for all $t \geq 0$.

Entrepreneurs' production and investment. Appendix A.1 shows that profit maximization by entrepreneurs when setting the prices of intermediate goods implies that the real wage w is:

$$w = \frac{\alpha(\epsilon - 1)}{\epsilon} = \alpha(1 - \mu), \tag{15}$$

where

$$\mu \equiv \frac{1}{\epsilon} \tag{16}$$

is entrepreneurs' mark-up—real profit per unit of output.

Entrepreneurs invest $I = 0$ in their investment technology if $r > \rho$, and $I = +\infty$ if $r \leq \lambda\rho$. For $r \in (\lambda\rho, \rho)$ they invest I such that both their incentive-compatibility constraint and the participation constraint of households bind.[9] Incentive compatibility requires that they hold a stake larger than $1 - \lambda$ in their projects, and so investment size I solves

$$\frac{\lambda\rho I}{r} = I - (\mu Y - f), \tag{17}$$

where $Y = \alpha L$ is entrepreneurs' (and aggregate) output. Condition (17) states that the funds $I - (\mu Y - f)$ borrowed from households by entrepreneurs—equal to total investment I minus entrepreneurs' own resources $\mu Y - f$—must be equal to the pledgeable part of the investment's payoff $\lambda\rho I$ discounted at r.

Bond market clearing. Bond-market clearing then yields the real rate r^*. The central bank has a zero-net supply of bonds. The net bond demand of the private sector is equal to households and entrepreneurs' savings net of their investment in entrepreneurs' storage technology. The private sector's savings are simple functions of the real rate r. Combining (10), (11), and (15) implies that households' real savings are

$$\frac{B_t}{P_t} = wL - C^Y = \delta\frac{1-\mu}{\mu}r - \phi(\beta r) \tag{18}$$

and entrepreneurs' real profit from production is

$$\mu Y - f = \delta r - f, \tag{19}$$

where

$$\delta \equiv \frac{\alpha^2\beta}{\gamma}\mu(1-\mu), \phi \equiv (u')^{-1}. \tag{20}$$

This implies that private savings as a function of r are

$$S(r) \equiv \frac{\delta}{\mu}r - \phi(\beta r) - f. \tag{21}$$

Injecting (19) in (17) yields in turn investment $I(r)$ as a function of r over$(\lambda\rho, \rho)$:

$$I(r) \equiv \frac{(\delta r - f)r}{r - \lambda\rho}. \tag{22}$$

The equilibrium real rate r^* is thus defined as follows:

[9] Appendix A.1 details the solution to this standard optimal-contracting problem.

1. If the (unique) solution r^* to $S(r) = 0$ is such that $r^* > \rho$ then it is the real rate, $I = 0$, and entrepreneurs lend their profits to households.
2. Otherwise, if $I(\rho) > S(\rho)$, then $r^* = \rho$ and $I \in [0, I(\rho)]$ is such that $I = \delta\rho/\mu - \phi(\beta\rho) - f$.
3. Otherwise, $r^* \in (\rho\lambda, \rho)$, and the real rate r^* is the unique solution to $S(r) - I(r) = 0$.

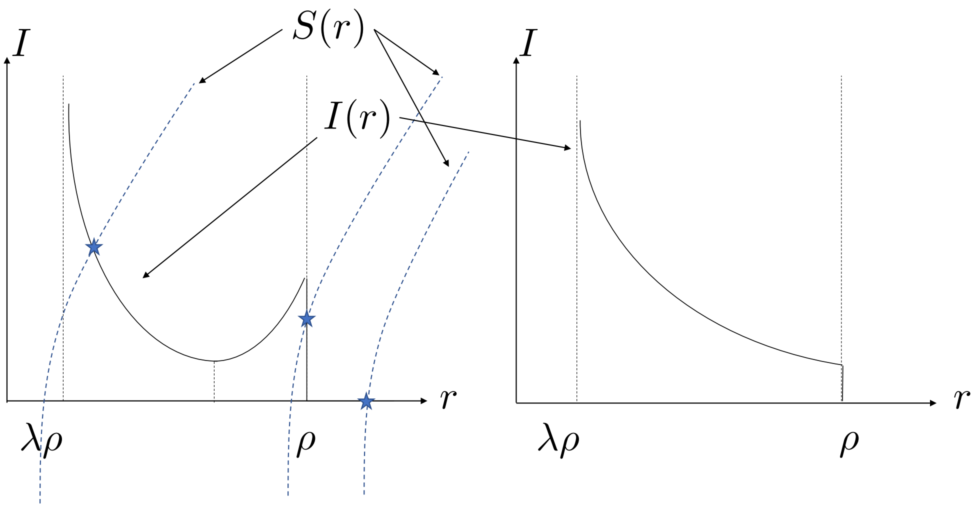

Figure 1: Investment and interest rate in the non-bubbly equilibrium

Figure 1 depicts in the plane (r, I) the graphs of $I(r)$ and $S(r)$ and their intersections at the equilibirum real rate. The function $I(r)$ may be either decreasing, or decreasing then increasing depending on parameter values, and we depict both cases.[10] The reason is that two forces compete in shaping entrepreneurs' investment capacity $I(r)$. This capacity is driven both by their net worth $\delta r - f$ and by the leverage ratio $r/(r - \lambda\rho)$ that applies to it. As r increases, so does their net worth. An increase in r also negatively affects their leverage ratio as they can finance with external funds a fraction at most equal to $\lambda\rho/r$, decreasing in r, of the proceeds from investment. The net-worth effect may prevail for sufficiently high rates because the leverage effect is marginally decreasing as r increases.

Relationship to Farhi and Tirole (2012a). Farhi and Tirole (2012a) exhibit the same tension between a negative leverage effect and a positive net-worth effect of an increase in the interest rate

[10] It increases after some threshold interest rate if, for example, the pledgeable fraction of investment λ is sufficiently small *ceteris paribus*. Appendix A.1 shows that S and I have a unique intersection either way.

on the demand for funds of constrained entrepreneurs. In their setup, entrepreneurs' net worth increases with respect to the interest rate because they must by assumption store their exogenous endowment at this rate before coming across an investment opportunity. We do not need this friction here. Entrepreneurs' endogenous endowment increases with respect to the interest rate because a higher rate spurs labor supply by young households enjoying a higher return on saved earnings.

Equilibrium existence. The above analysis shows that there is at most one non-bubbly flexible equilibrium, and fully characterizes it when it exists, which is obviously the case if $f = 0$. Appendix A.1 establishes conditions under which this flexible equilibrium exists stated in the following proposition. Let $r^{*,0}$ denote the equilibrium real interest rate if $f = 0$.

Proposition 1. ***(Non-bubbly flexible-price equilibrium)*** *The equilibrium exists if and only if*

$$f \leq \delta r^* \left[1 - \frac{(\Pi^*)^{\frac{1-\mu}{\mu}}}{\mu} [1 - (1-\mu)\Pi^*]^+ \right], \tag{23}$$

with $\Pi^* = \Pi^M (r^*/r^M)^{1/\psi}$. *In particular, if* $\Pi^M (r^{*,0}/r^M)^{1/\psi} > 1$, *there exists* $\bar{f} > 0$ *such that* (23) *holds if and only if* $f \leq \bar{f}$.

When it exists, the flexible equilibrium is as follows. If $r^* < \rho$, *investment is constrained. If* $r^* > \rho$, *entrepreneurs lend to households rather than invest. It may also be that* $r^* = \rho$ *and investment is not constrained. Output* $\alpha L = \delta/\mu r^*$, *interest rate* r^*, *and investment* I *increase with respect to* λ *other things being equal.*

Proof. See Appendix A.1. □

Throughout the paper, we deem "output" the proceeds αL from applying labor L to entrepreneurs' production technology. To be sure, the total GDP per period in this economy—consumption by all agents plus investment—is equal to $\alpha L + \rho I$. It also increases with respect to λ from the proposition. As the pledgeability λ of entrepreneurs' ventures decreases, this reduces their ability to lever up their net wealth, which both reduces their investment capacity and raises the price of storage vehicles (depresses the interest rate) as the supply of such vehicles by entrepreneurs shrinks. Lower returns on savings in turn reduce life-long returns from supplying labor which depresses output.

The right-hand side of condition (23) is equal to 0 for $\Pi^* = 1$ and so the equilibrium exists only if $f = 0$ in this case: In the absence of inflation, there is no point adjusting prices in this perfect-foresight model. Since r^* and thus $\Pi^* = \Pi^M (r^*/r^M)^{1/\psi}$ depend on f, condition (23) does not explicitly define the sets of menu costs for which the equilibrium exists. If, however, $\Pi^M (r^{*,0}/r^M)^{1/\psi} > 1$, then this set is of the form $[0, \bar{f}]$ holding all other parameters fixed.

Do reasonable values of r^M and Π^M warrant the existence of this flexible equilibrium? There are two (at least qualitatively) plausible sets of values for the parameters of the Taylor rule Π^M and r^M for which the flexible equilibrium exists when the menu cost f is strictly positive but sufficiently smalll other things being equal. First, it can be the case that monetary policy targets the actual flexible-equilibrium real rate—$r^M = r^*$, and has a strictly positive inflation target $\Pi^M > 1$. Such a strictly positive inflation target is clearly in line with actual monetary policies. To be sure, there is however no case for a non-zero inflation target in this model, and very little more generally in modern models of monetary transmission (see Schmitt-Grohé and Uribe (2010) for a survey on this question). Coibion et al. (2012) find an optimal strictly positive (but small) inflation target in the presence of a zero lower bound. Incorporating this ingredient here (or presumably any other benefit from a positive target) would require to stray away from a perfect-foresight model.

The other situation in which the flexible equilibrium can be sustained is that in which the inflation target is $\Pi^M = 1$ but in which the rule targets a real rate r^M strictly smaller than the flexible-equilibrium rate r^*. This is a plausible target in light of the results below that given such a rule, not only the flexible equilibrium but also equilibria with lower inflation and real rates may be sustained. It is thus reasonable that the policy rate r^M be smaller than the highest value r^* that can be generated across equilibria. Section 3.1 will actually exhibit a sunspot equilibrium with stochastic regime changes leading to sunspot fluctuations of the real rate and realized inflation. The monetary authority facing data generated for example by this equilibrium would presumably set $r^M < r^*$.

In sum, both an inflation target Π^M sufficiently large and a rate r^M sufficiently low help sustain the flexible equilibrium. The former condition is realistic but can only be justified outside this very simple model, whereas the latter arises more naturally as a natural consequence from the multiplicity of equilibria.

2.2 "Natural" bubbles

This section discusses bubbles in flexible-price equilibria. Its goal is not to exhaustively describe such bubbly equilibria. It rather seeks to illustrate some properties of bubbles when prices are flexible that stand in stark contrast with the properties of bubbles in the presence of price rigidity studied below in Section 3.

Suppose that there exists a flexible-price non-bubbly equilibrium such that $r^* < 1$ and $\Pi^* > 1$. There clearly exist parameters such that this is the case provided $\lambda\rho < 1$. Define

$$B = S(1) - I(1). \tag{24}$$

$B > 0$ since $I(r^*) = S(r^*)$ and $r^* < 1$. We have:

Proposition 2. ***(Natural bubbles, inflation, and investment)***

- *For every $b \in (0, B]$ there exists a bubbly equilibrium whereby households trade a bubble with date-0 value b.*
- *These bubbly equilibria display at each date a higher inflation than the non-bubbly one, increasing in b.*
- *Households' utility is higher in the presence of these bubbles at each date whereas that of entrepreneurs may or may not be higher.*

Proof. See Appendix A.2. □

The remainder of the paper deems "natural" these bubbles that may arise because the interest rate r^* (when prices are flexible) is smaller than one, as opposed to the policy-induced bubbles studied in Section 3 that will grow when prices are not flexible.

To simply illustrate these results, consider the bubbly steady state: the (unique) situation in which the bubble of constant size B is first sold by old households at date 0 and then perpetually refinanced at a unit interest rate. The right-hand panel in Figure 2 shows bubble size B as the wedge between savings and investment at $r = 1$.

Comparing investment at bubbly and non-bubbly steady states, Figure 2 shows that the rise of the bubble always crowds investment out when $I(r)$ is decreasing but may but crowd it in when it has an increasing portion.[11] The intuition is simply that the increase in interest rate caused by the bubble has a negative impact on entrepreneurs' leverage ratio that may or may not offset the increase in their own resources.

[11] The left-hand panel in Figure 2 illustrates crowding out when the non-bubbly steady state is at E and crowding in when it is at E'.

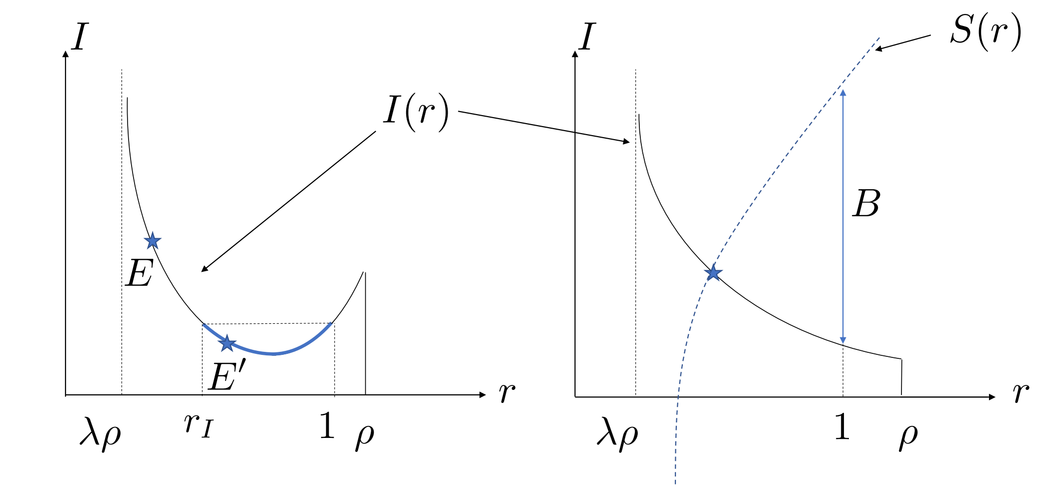

Figure 2: Investment in the bubbly steady state

The proof of Proposition 2 shows that when it occurs, this increase in investment may more than offset that of the interest rate, thereby leading to an increase in entrepreneurs' utility.

The important results in this section is that the bubbles described in Proposition 2 can be permanently good for investment and for entrepreneurs' welfare, and that their size is positively correlated with inflation across equilibria.

3 Equilibria with price rigidity

This section studies equilibria in which entrepreneurs do not always adjust their prices. Section 3.1 first studies equilibria in which prices are fixed. To be sure, this is an unrealistic polar case. Many ingredients missing in this model (entry, exit, new products, aggregation and dissemination of private information) would lead to price flexibility in the longer run. I still detail this case because it delivers the important insights in the simplest setting. Proposition 5 exhibits equilibria with more realistic forms of price rigidity, that are limited in time or/and in severity. Section 3.2 studies bubbles in these rigid-price environments.

3.1 Fixed-price equilibrium

I first characterize a (non-bubbly) fixed-price equilibrium and then discuss its existence. Suppose thus that for all $i \in [0,1]$ and $t \geq 0$, $P_t^i = P_t = P_{-1}$ and $\Pi_t = 1$. The equilibrium must then be as follows.

Optimal supply of labor and savings by households. Denoting W_t the nominal wage, each date-t household selects a nominal investment in bonds B_t and a labor supply L_t that solve:

$$\max_{B_t, L_t} u(C_t^Y) + \beta C_t^O - \frac{\gamma L_t^2}{2} \tag{25}$$

s.t.

$$P_{-1} C_t^Y + B_t \leq W_t L_t, \tag{26}$$

$$P_{-1} C_t^O \leq R_t B_t + P_{-1} e, \tag{27}$$

$$C_t^Y, C_t^O, L_t \geq 0, \tag{28}$$

Optimal labor supply yields

$$W_t u'(C_t^Y) = \gamma P_{-1} L_t, \tag{29}$$

and optimal bond investment yields

$$u'(C_t^Y) = \beta R_t \frac{P_{-1}}{P_{-1}} = \beta R_t. \tag{30}$$

Determination of the real rate. The Fisher equation (30) combined with the Taylor rule (3) yields for all $t \geq 0$

$$\frac{u'(C_t^Y)}{\beta} = R_t = r^M \Pi^M \left(\frac{1}{\Pi^M} \right)^{1+\psi}, \tag{31}$$

which pins down the time-invariant real rate $u'(C^Y)/\beta$ that we denote $\hat{r}$:

$$\hat{r} = \frac{r^M}{(\Pi^M)^\psi}. \tag{32}$$

Optimal investment by entrepreneurs. For brevity, I restrict the analysis to the case in which $\hat{r} < \rho$. In this case, entrepreneurs borrow up to their constraint:

$$I_t = \frac{(\alpha - w_t) L_t \hat{r}}{\hat{r} - \rho\lambda}, \tag{33}$$

The zero bond supply of the central bank implies that investment must be equal to households and entrepreneurs' savings for the bond market to clear:

$$w_t L_t - C_t^Y + (\alpha - w_t) L_t = I_t. \tag{34}$$

Households' consumption when young C_t^Y, labor supply L_t, investment I_t, and the real wage $w_t = W_t/P_{-1}$ thus obey four equations {(29);(30); (33);(34)}. The proof of Proposition 3 shows that this system admits a unique (obviously time-invariant) solution.

Two steps are left to show that this defines a fixed-price equilibrium. First, it remains to prove that entrepreneurs are willing to accommodate demand at these fixed prices. The proof of Proposition 3 shows that this is so if $\hat{r} > \underline{r}$, where.

$$\underline{r} \equiv \max\left\{\lambda\rho; \inf\left\{x \mid \frac{\alpha^2\beta x}{\gamma} \geq \phi(\beta x)\right\}\right\}. \tag{35}$$

Second, it must be that entrepreneurs prefer to accommodate demand at fixed prices rather than adjust their prices, which is obviously true as soon as f is sufficiently large holding all other parameters fixed. The following proposition summarizes these results.

Proposition 3. ***(Non-bubbly equilibrium with fixed prices)*** *Suppose $r^M/(\Pi^M)^\psi \in (\underline{r}, \rho)$. If f is sufficiently large, there exists a unique non-bubbly equilibrium with fixed prices. The real rate is $\hat{r} = r^M/(\Pi^M)^\psi$.*

The real quantities $(C^Y, L, I, w, \hat{r})$ that fully characterize the equilibrium are identical to the ones that would obtain in the flexible-price equilibrium of an economy with the same parameters $(\alpha, \rho, \lambda, \beta, \gamma, u(.))$, but with $f = 0$ and a different value of $\mu = 1/\epsilon$.

Proof. See Appendix A.3. □

When deriving the real block (quantities and relative prices) of the fixed-price equilibrium relative to the flexible one, one first-order condition is missing—profit maximization by entrepreneurs. Thus, unlike in the flexible case, monetary policy contributes to characterizing this real block. The assumption of a Taylor rule implies that monetary policy does so here by selecting the real rate.

Proposition 3 states that this real block of the fixed-price equilibrium is formally identical to that of the flexible-price equilibrium in an economy in which $f = 0$, and the real parameters are otherwise identical except for $\mu = 1/\epsilon$. Let us denote $\mu(\hat{r})$ the implicit markup associated this way with the real rate $\hat{r} = r^M/(\Pi^M)^\psi$.

That this fixed-price equilibrium exists if it is sufficiently costly to adjust prices is neither surprising nor interesting. The following proposition offers conditions under which, much more interestingly, both the flexible-price equilibrium in Proposition 1 and this fixed-price one exist.

Proposition 4. ***(Multiple equilibria)*** *Both the flexible-price equilibrium in Proposition 1 and the fixed-price equilibrium in Proposition 3 can be sustained if and only if*

$$\hat{r}\left[\left(\frac{1-\mu}{1-\mu(\hat{r})}\right)^{\frac{1-2\mu}{\mu}} - \frac{\mu(\hat{r})(1-\mu(\hat{r}))}{\mu(1-\mu)}\right] \leq \frac{f}{\delta} \leq r^*\left[1 - \frac{(\Pi^*)^{\frac{1-\mu}{\mu}}}{\mu}[1-(1-\mu)\Pi^*]^+\right]. \tag{36}$$

Fix the real parameters of the economy $(\alpha, \rho, \lambda, \beta, \gamma, u(.), \mu, f)$*, and* r^M*. Suppose that there exists* $x \in (\underline{r}, \rho)$ *and* $y > 0$ *such that* (36) *holds with* $\hat{r} = x$ *and* $\Pi^* = y$*. Then there exists* Π^M *and* ψ *such that both the flexible-price and fixed-price equilibria can be sustained.*

Fix all the parameters of the economy but f*,* Π^M*, and* ψ*. These three parameters can be chosen such that both flexible-price and fixed-price equilibria can be sustained.*

Proof. See Appendix A.4. □

Proposition 4 shows that it is not difficult to find parameters that support both the flexible and fixed price equilibria. Besides stating condition (36), the proposition illustrates it in two ways. Notice that the monetary parameters (r^M, Π^M, ψ) affect only $\hat{r}$ and Π^* in condition (36). The proposition first states that if the real parameters of the model are such that both equilibria exist for some values of $\hat{r}$ and Π^*, then two of the monetary parameters, e.g, Π^M and ψ, can be chosen to reach them regardless of the value of the third. It then states that using only one real parameter as a degree of freedom, e.g., f, one can ensure that the real parameters satisfy this condition.

This multiplicity of equilibria given a Taylor rule (3) is a central ingredient of the paper. It contrasts with the situation that arises with other forms of nominal rigidities such as Calvo pricing. The fixed menu cost here opens up the possibility of a multiplicity of joint values of the real rate and inflation that can be supported in equilibrium given an announced Taylor rule. The following proposition exhibits more equilibria that fall in between these polar cases of full flexibility and full rigidity.

Proposition 5. ***(More realistic price rigidities)*** *Suppose condition* (36) *holds with strict inequalities.*

- *There exists* $p \in (0, 1)$ *such that the economy starts out with rigid prices and snaps back to flexible prices forever with probability* p *at each date.*
- *There exists* $x \in (0, 1)$ *and an initial distribution of prices such that there exists a steady state in which a fraction* x *of entrepreneurs adjust their prices at each date. The real rate and inflation rate are both in between their values in the flexible and fixed price equilibria.*

Proof. See Appendix A.5. □

The stochastic equilibrium classically shows that the forces leading to multiple perfect-foresight equilibria can also lead to a purely endogenous form of uncertainty in equilibrium.

The steady state with staggered price adjustments is similar to that in Caplin and Spulber (1987). In this equilibrium, at each date, the population of entrepreneurs that has not updated its price since $\lfloor 1/x \rfloor + 1$ dates does it, and a mass $x - 1/(\lfloor 1/x \rfloor + 1)$ of the population that has done it exactly $\lfloor 1/x \rfloor$ dates ago does it, where $\lfloor 1/x \rfloor$ is the integer part of $1/x$. The rest of the population does not adjust its price. Notice that for the economy to be in this steady state from date 0 on, it must start out with the steady-state distribution of prices instead of the degenerate one assumed in the paper for simplicity.

In the cross section of the three deterministic steady states—flexible, infrequently adjusted, and fixed prices, as the frequency of price adjustments decreases, so do the level of inflation and the real rate of interest, whereas the wage increases, even though all equilibria correspond to the same economic fundamentals and to the same monetary policy. The intuitition is as follows. Given a real rate r', inflation must satisfy

$$\frac{\Pi_{t+1}}{\Pi_t} = \frac{r^M}{r'} \left(\frac{\Pi_t}{\Pi^M} \right)^{\psi}, \tag{37}$$

and $\Pi_t = \Pi' = \Pi^M (r'/r^M)^{1/\psi}$ for all t is the only non-exploding path. Thus the Taylor rule imposes this correlation between real rate and inflation across equilibria. That the equilibria with lower inflation correspond to more price rigidity then follows from the fact that it is all the more valuable to adjust one's price because inflation is high and one's relative price gets far from optimal.

Can the monetary authority select equilibria? In the presence of a fixed price-setting cost, the monetary authority does not pin down a unique equilibrium by committing to a baseline interest-feedback rule. This raises the question whether it can select or equivalently eliminate equilibria by committing to a more sophisticated rule. I conjecture it does. Suppose for example that the monetary authority perfectly observes not only the realized inflation at each date but also the fraction of entrepreneurs who pay the menu cost. It can then make its rule contingent on this latter variable as well. This entails that it can eliminate equilibria with a given degree of price flexibility by making sure that they are not sustainable given the rule. For example, a contingent rule ensuring that $\Pi^* = 1$ in the fully flexible equilibrium destroys this equilibrium because adjusting prices cannot be optimal in equilibrium in this case. Similarly, the fixed-price equilibrium can be eliminated by committing to an $\hat{r}$ such that the left-hand inequality in (36) cannot be satisfied when entrepreneurs do not adjust prices. A full-fledged study of such more sophisticated rules is an interesting route for future research. A fair assessment of their merits should in particular take into account that their inputs can only be observed with noise in practice, thereby leading to policy mistakes.

3.2 Policy-induced bubbles

This section studies the properties of bubbles in the presence of fixed prices. I deem "policy-induced bubbles" such bubbles because the condition $\hat{r} \leq 1$ that makes them possible depends only on monetary policy. Suppose that the fixed-price non-bubbly equilibrium in Proposition 3 exists and is such that $\hat{r} \leq 1$. I study fixed-price equilibria with bubbles in two steps for expositional simplicity. Subsection 3.2.1 first studies bubbly equilibria in which, from $t = 0$ on, the same bubble is sold by old date-t households to young date-t households. This perpetual rollover of a legacy bubble is the typical situation studied in the literature on rational bubbles. This is also the one on which Farhi and Tirole (2012a) focus their analysis. Borrowing from Martin and Ventura (2012), I deem these situations in which new bubbles are never issued ones of "old bubbles". Subsection 3.2.2 then studies more general bubbly equilibria, among them those in which young entrepreneurs may sell new bubbles, a situation that I deem one of "new bubbles" following Martin and Ventura (2012) again.

I assume throughout this section that f is sufficiently large that entrepreneurs never find it optimal to adjust their prices in the presence of a bubble no matter its size.

3.2.1 Old bubbles

Adding time dependence to our respective notations C^Y, L, and w for young households' consumption and labor supply, and for the real wage respectively, a fixed-price equilibrium with an old bubble with date-t value b_t is characterized by a sequence $(b_t, C_t^Y, L_t, w_t)_{t\in\mathbb{N}} \in ([0, +\infty)^4)^{\mathbb{N}}$ that satisfies:

(38) $$w_t u'(C_t^Y) = \gamma L_t,$$

(39) $$u'(C_t^Y) = \beta\hat{r},$$

(40) $$\alpha L_t - C_t^Y - b_t = \frac{(\alpha - w_t)L_t\hat{r}}{\hat{r} - \rho\lambda}.$$

(41) $$b_0 > 0, b_{t+1} = \hat{r}b_t,$$

(42) $$w_t \leq \alpha.$$

Conditions {(38);(39)} state that households optimally supply labor and capital. Condition (40) states that savings net of the bubble are equal to investment. Conditions (41) state that the bubble exists and that households are willing to roll it over, and (42) ensures that firms are willing to accommodate demand. The following proposition characterizes equilibria with old bubbles. Let

(43) $$b^{max} = \frac{\alpha^2\beta\hat{r}}{\gamma} - \phi(\beta\hat{r}).$$

Proposition 6. ***(Old policy-induced bubbles are always bad for investment and entrepreneurs)*** *For each $b_0 \in (0, b^{max})$, there exists a unique equilibrium with an old bubble with initial value b_0. There exists no equilibrium with an old bubble with initial value larger than b^{max}. In such equilibria, bubbles earn an expected return $\hat{r}$. Output and households' utility at every date are higher* *than in the non-bubbly fixed-price equilibrium and increasing in b_0 across equilibria, whereas investment and entrepreneurs' utility at every date are lower than in the non-bubbly fixed-price equilibrium, and decreasing in b_0 across equilibria.*

Proof. See Appendix A.6. □

Comparing these equilibria with the bubbly steady state in the flexible-price model in Section 2.2 shows two major differences between the natural bubbles that arise in the flexible-price case and these policy-induced ones when prices are fixed:

1. Natural bubbles raise the interest rate whereas policy-induced bubbles do not affect it since the monetary authority controls it. Policy-induced bubbles thus earn low returns themselves.
2. Whereas natural bubbles may be either perpetual substitute or complement to investment, and either good or bad for entrepreneurs (Proposition 2), a policy-induced bubble, once issued, always crowds out investment and reduces entrepreneurs' utility.

A useful way to compare natural and policy-induced bubbles consist in studying their respective impacts on prices and quantities in the capital market. Consider first natural bubbles. Equilibrium in the capital market in the flexible-price non-bubbly equilibrium implies:

(44) $$\frac{(\delta(\mu)r - f)r}{r - \lambda\rho} = \frac{\delta(\mu)r}{\mu} - \phi(\beta r) - f,$$

where $\delta(x) = \alpha^2\beta x(1-x)/\gamma$ is increasing over $(0, 1/2]$ whereas $\delta(x)/x$ is decreasing. The left-hand side of (44) is entrepreneurs' investment and the right-hand one is aggregate savings. The presence of a natural bubble b leaves μ of course unchanged but affects the equilibrium interest rate, which jumps to a value $1 \geq r' > r$ such that

(45) $$\frac{(\delta(\mu)r' - f)r'}{r' - \lambda\rho} = \frac{\delta(\mu)r'}{\mu} - \phi(\beta r') - b - f.$$

As seen in Section 2, labor and capital share both increase in the presence of a bubble. Investment may or may not increase depending on whether the leverage effect more than offsets this.

Consider then policy-induced bubbles. Equilibrium in the capital market in the fixed-price non-bubbly equilibrium implies:

(46) $$\frac{\delta(\mu(\hat{r}))\hat{r}^2}{\hat{r} - \lambda\rho} = \frac{\delta(\mu(\hat{r}))\hat{r}}{\mu(\hat{r})} - \phi(\beta\hat{r}).$$

Unlike in the flexible-price case, the presence of a bubble b now leaves the interest rate $\hat{r}$ unchanged. The proof of Proposition 6 shows that the real block of an equilibrium with an old bubble

b at some date t is isomorphic to that of a flexible-price economy with a natural bubble b, identical parameters $(\alpha, \rho, \lambda, \beta, \gamma, u(.))$, and a markup $\mu^b < \mu(\hat{r})$ and decreasing in b:

$$\frac{\delta(\mu^b)\hat{r}^2}{\hat{r} - \lambda\rho} = \frac{\delta(\mu^b)\hat{r}}{\mu^b} - \phi(\beta\hat{r}) - b. \tag{47}$$

The proof of Proposition 6 shows that this entails in turn that output and the labor share increase with a bubble $b > 0$, whereas the capital share and investment must be lower in the presence of the bubble than in its absence. Intuitively, it is possible to squeeze bubbles on top of investment projects only if households overall have more investable funds. This must come at a reduction in the capital share relative to the non-bubbly equilibrium. This reduction in entrepreneurs' net wealth always negatively affects investment.

In sum, a compact way of stating the difference between natural and policy-induced bubbles is that the former affect r whereas the latter acts as if it was affecting μ, and this shapes their respective impacts on the economy.

3.2.2 New bubbles

Policy-induced bubbles can never boost investment once they have been issued because, unlike natural bubbles, they always reduce the profits that entrepreneurs can lever up. Similarly, if old agents (households or entrepreneurs) issue and sell new policy-induced bubbles at a given date, then it is easy to see that these new bubbles also negatively affect investment upon issuance.[12]

The case in which new policy-induced bubbles are issued by young entrepreneurs is different, however. Intuitively, these bubbles still drain profits out of entrepreneurs so that households can purchase them. Yet the proceeds from selling these bubbles boost entrepreneurs' investable funds and thus investment capacity. Such new bubbles issued by young entrepreneurs thus may boost investment upon issuance, as do that issued by the most efficient producers of capital goods in Martin and Ventura (2012). This section studies such new policy-induced bubbles issued by young entrepreneurs. A fixed-price equilibrium in which young entrepreneurs issue a fraction $\omega_t \in [0, 1]$

[12]Equilibria with such new bubbles are characterized by equations $\{(38);(39);(40);(41);(42)\}$ up to the only modification in (41) that $b_{t+1} = \hat{r}b_t$ is replaced with $b_{t+1} \geq \hat{r}b_t$, which does not affect the proof of welfare results in Proposition 6.

of the total bubble b_t sold at date t is a sequence $(\omega_t, b_t, C_t^Y, L_t, w_t)_{t\in\mathbb{N}} \in ([0,+\infty)^5)^{\mathbb{N}}$ that satisfies:

$$w_t u'(C_t^Y) = \gamma L_t, \tag{48}$$

$$u'(C_t^Y) = \beta\hat{r}, \tag{49}$$

$$\alpha L_t - C_t^Y - (1-\omega_t)b_t = \frac{[(\alpha - w_t)L_t + \omega_t b_t]\hat{r}}{\hat{r} - \rho\lambda}. \tag{50}$$

$$\omega_t \in [0,1], (1-\omega_{t+1})b_{t+1} \geq \hat{r}b_t, \tag{51}$$

$$w_t \leq \alpha. \tag{52}$$

There are two differences with the above characterization of old bubbles. First, the inequality in (51) states that the fraction $1-\omega_{t+1}$ of the date-$(t+1)$ bubble b_{t+1} that does not correspond to new bubbles issued by date-$(t+1)$ young entrepreneurs must pay for the sale of legacy bubbles worth $\hat{r}b_t$, and possibly for new bubbles issued by old agents. Second and more important, equilibrium in the capital market (50) now encodes that only a fraction $1-\omega_t$ of the savings that go into bubbles does not fund investment (left-hand side), and that a fraction ω_t of them accrues to young entrepreneurs' net worth (right-hand side). The following proposition offers sufficient conditions for positive and negative impacts of new bubbles on investment.

Proposition 7. ***(New policy-induced bubbles and investment)***
In a bubbly fixed-price equilibrium characterized by $\{$(48)*;*(49)*;*(50)*;*(51)*;*(52)$\}$*:*

- *Date-t investment is smaller than in the non-bubbly fixed-price equilibrium if* $\omega_t \leq (1-2\mu(\hat{r}))/[2(1-\mu(\hat{r}))]$*;*
- *Date-t investment is larger than in the non-bubbly fixed-price equilibrium if* $\omega_t \geq 1/2$*;*
- *Otherwise, date-t investment is larger than in the non-bubbly fixed-price equilibrium if b_t is below a threshold (that depends on ω_t).*
- *If* $\hat{r} > 1/[2(1-\mu(\hat{r}))]$*, investment in this equilibrium cannot exceed that in the non-bubbly one at every date.*

Proof. See Appendix A.7. □

Proposition 7 shows that a bubbly equilibrium can be associated with more investment than the non-bubbly one at a given date t if the fraction ω_t of the total date-t bubble corresponding to new bubbles issued by young entrepreneurs is at least 50%, or if it is at least $(1-2\mu(\hat{r}))/[2(1-\mu(\hat{r}))]$ and the total bubble is not too large.

Overall, the first three points in Proposition 7 imply that if a large policy-induced bubble has a large positive impact on investment at a given date, then it will become contractionary for investment soon after provided $\hat{r}$ has plausible values (i.e., is below but close to one) because it is not possible to repeatedly issue a new bubble sufficiently large to boost investment given such a rate $\hat{r}$ at which legacy bubbles are refinanced.

On a related note and more formally, the last point in Proposition 7 shows that even when allowing for arbitrary patterns of new bubbles, it is impossible that policy-induced bubbles boost investment at every date for plausible parameter values.[13] This contrasts with the case of natural bubbles, for which Section 2.2 exhibits a simple example of a bubble that is a perpetual complement to investment.

Scope of the results on bubbles. In sum, policy-induced bubbles cannot be as favorable to investment as natural ones, even in the best cases in which young entrepreneurs can issue them. It is important to stress that in our environment, as in that set by Farhi and Tirole (2012a) and Martin and Ventura (2012), agents are short-lived, and in particular do not switch types over time. This presumably plays an important role in generating our stark results. Policy-induced bubbles could be more favorable to investment in an environment in which households could become entrepreneurs later on in their life, and lever up the accumulated savings from the higher wages that they earned in the presence of a bubble when young. An interesting route for future research consists in studying how the rise of a bubble—natural or policy-induced—affects the dynamics of investment over time in a more general model of wealth accumulation by entrepreneurs.

4 Inflation versus policy-induced bubbles

This section applies the results in Section 3 to the study of two economies that display interesting feedbacks between CPI inflation and asset bubbles. The first economy is one in which bubbles can arise in equilibrium even though the real rate in the flexible equilibrium is larger than the (unit) growth rate of the economy. In the second economy, a binding lower-bound constraint determines the official rate. An interesting insight is that in both economies, and for different reasons, CPI inflation and asset bubbles are incompatible equilibrium phenomena in the sense that they do not occur jointly. This contrasts with natural bubbles that tend to arise together with higher CPI inflation, as seen in Section 2.2.

[13]Condition $\hat{r} > 1/[2(1-\mu(\hat{r}))$ holds as soon as the gross rate $\hat{r}$ exceeds 0.9 and the labor share in the fixed-price model $1-\mu(\hat{r})$ exceeds 56%.

4.1 Inflation and bubbles as competing monetary phenomena

Consider an economy such that both the flexible and fixed price equilibria exist, with

$$\hat{r} < 1 < r^*, \tag{53}$$

and condition (36) holds strictly. Proposition 4 warrants the existence of such an economy.

Proposition 8. ***(Inflation and bubbles as competing monetary phenomena)*** *In this economy:*

1. *Any perfect-foresight equilibrium such that*

$$\Pi^l \equiv \lim_{T\to+\infty} \left(\prod_{t=0}^{T} \Pi_t \right)^{\frac{1}{T}} \tag{54}$$

   exists must be such that $\Pi^l \leq \Pi^M/(r^M)^{1/\psi} = \Pi^*/(r^*)^{1/\psi} < \Pi^*$ *if it features a bubble.*

2. *There exist stochastic equilibria comprised of the two following phases. They start out with fixed prices and the growth of a policy-induced bubble. At the random date at which this first phase ends, the bubble bursts, the economy reverts back to the flexible-price equilibrium, and sticks to it forever. Inflation picks up and both real and nominal interest rates increase.*

Proof. See Appendix A.8. □

Proposition 8 illustrates in two ways the idea that policy-induced bubbles and high CPI inflation are incompatible in the sense that they do not jointly occur in equilibrium.

The first point offers a general formulation of this tension between bubbles and inflation across all perfect-foresight equilibria. It states that inflation in all equilibria that feature a bubble, however small, is bounded away from the maximum inflation Π^* in the flexible equilibrium—in which bubbles cannot arise. The intuition is simply that a bubble requires sufficiently low interest rates, and that this corresponds to equilibria with low inflation as well.

The class of simple stochastic equilibria described in the second point of the proposition merely adds a bubble to the stochastic equilibria described in Proposition 5. A policy-induced bubble rises and bursts in an economy in which the long-run real rate is larger than one. Inflation contemporaneous to this bubble is low and picks up as it bursts. Notice that if these stochastic bubbles were attached to a particular asset or asset class, they would amplify the impact of the variations of the interest rate on its valuation, as they burst right when the real rate increases.[14] Such monetary bubbles that magnify the effect of monetary policy on asset prices may contribute to the impact of the stance of monetary policy on asset valuation for which Bianchi et al. (2022) recently find empirical support.

[14]It would be straightforward to add a "tree" to which bubbles are attached, as in Tirole (1985).

4.2 At the zero lower bound

Given the stated goal of exhibiting bubbles in an economy in which they could not be sustained in the absence of monetary frictions, the economy in the previous section 4.1 is deliberately such that the real rate, when determined by monetary policy, is below the flexible-price one ($\hat{r} < 1 < r^*$). Yet the dominant view is that a binding lower bound has been an important feature of monetary policy over the last decade or so. Accordingly, to address this situation, this section studies the opposite case in which the real rate when set by monetary policy is above the natural one because of a binding lower-bound constraint. Suppose thus that the monetary authority must keep the nominal rate above a lower bound $\eta \in (0, 1]$. We have:

Proposition 9. ***(At the lower bound, no price rigidity without monetary bubbles)*** *Suppose parameters are such that $r^*, \hat{r} < \eta$. Then there exist a range of menu costs f such that any rigid-price equilibrium features a monetary bubble.*

Proposition 9 exhibits an interesting feature of our economy at the zero lower bound: The feedback between policy-induced bubbles and price rigidity may now go in both directions. More precisely, policy-induced bubbles always need the economy to be in the fixed-price equilibrium to arise: This is how we define them. Interestingly, when the official rate is kept above the natural one by a lower-bound constraint, the causality may also go the other way: The fixed-price equilibrium is a sustainable outcome only in the presence of policy-induced bubbles.

The intuition is straightforward. When the lower-bound constraint binds, the rigid-price equilibrium, if sustainable, has the same real features as that of a flexible-price economy with a markup higher than the actual one μ leading to a lower real wage. This is deflationary as it makes it tempting for each entrepreneur to set its price below the equilibrium value P_{-1}. The introduction of a bubble reduces this markup, setting it closer to μ provided the bubble is not too large. Thus a policy-induced bubble by lifting the wage makes the statu-quo price closer to the profit-maximizing one. As a result, there exist a range of menu costs f such that optimizing prices is optimal in the absence of a bubble even when other firms do not adjust their prices, whereas it becomes suboptimal to do so in the presence of a bubble.

In sum, the situation in which $r^* > 1 > \hat{r}$ in Section 4.1 is such that there cannot be bubbles when prices are flexible, whereas that in which $r < \delta \leq 1$ in this Section 4.2 is such that prices cannot be rigid in the absence of bubbles, introducing another source of incompatibility between high inflation expectations and bubbles across equilibria.

5 Conclusion

The starting point of this paper is the insight in Ball and Romer (1991) that fixed menu costs may create multiple equilibria with varying price rigidity when prices are strategic complements. In their introduction, Ball and Romer highlight that their contribution integrates two important paradigms of Keynesian economics—multiple equilibria and nominal rigidities. This paper revisits this broad idea of self-justified nominal rigidities in an economy that features i) a standard interest-feedback rule, and ii) a combination of financial frictions and incompleteness (OLG) that paves the way to bubbles. This generates a multiplicity of equilibria across which the real interest rate, inflation, and price rigidity comove.

This enables us to rationalize the widespread narrative that an accommodative monetary policy may create not much else than froth in financial markets in the form of bubbles that crowd out investments with superior returns. Such bubbles as pure monetary phenomena starkly differ from natural ones in three interesting ways. First, they are compatible with an environment of low expected returns and earn low expected returns themselves regardless of their size. Second, they burst when CPI inflation picks up. Finally, unlike natural bubbles that may be either good or bad for investment, such policy-induced bubbles once issued always hurt the most productive but constrained agents of the economy by diverting resources away from them.

An interesting feature of the model is that such policy-induced bubbles are not an ineluctable consequence of monetary easing. They coexist with alternative equilibria that display more standard nominal and real effects of monetary policy. On the other hand, our approach suffers from the same limited predictive and normative power as does any theory relying on equilibrium multiplicity. The multiplicity of bubbly (or not) equilibria given low real rates is inherently difficult to reduce. We conjecture that the multiplicity along price rigidity may lend itself to iterated-dominance treatments such as (dynamic versions of) global games. It would be interesting to compare the comovement of price rigidity with other economic fluctuations in such a model with that in models in which sufficient heterogeneity across agents warrants equilibrium uniqueness in the presence of menu costs. We leave this route for future research.

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Appendix

A.1 Proof of Proposition 1

Optimal expenditures across intermediate goods. Optimal spending of a given nominal income X across date-t intermediate goods by a private agent (household or entrepreneur) reads:

$$\max_{(x_i)_{i\in[0,1]}} \left(\int_0^1 x_i^{1-\frac{1}{\epsilon}} di\right)^{\frac{\epsilon}{\epsilon-1}} \tag{A.1}$$

$$s.t. \int_0^1 P_t^i x_i di = X. \tag{A.2}$$

This yields a demand for good $i \in [0,1]$

$$x_i = x\left(\frac{P_t}{P_t^i}\right)^{\epsilon}, \tag{A.3}$$

where $P_t = (\int_0^1 P_t^{i\,1-\epsilon} di)^{1/1-\epsilon}$ and $x = X/P_t$.

Optimal supply of labor and savings by households. Denoting W_t the nominal wage, each date-t household selects a nominal investment in bonds B_t and a labor supply L_t that solve:

$$\max_{B_t, L_t} u(C_t^Y) + \beta C_t^O - \frac{\gamma L_t^2}{2} \tag{A.4}$$

s.t.

$$P_t C_t^Y + B_t \leq W_t L_t, \tag{A.5}$$

$$P_{t+1} C_t^O \leq R_t B_t + P_{t+1} e, \tag{A.6}$$

$$C_t^Y, C_t^O, L_t \geq 0, \tag{A.7}$$

where e is the household's exogenous endowment when old. Optimal labor supply yields

$$W_t u'(C_t^Y) = P_t \gamma L_t, \tag{A.8}$$

and optimal bond investment yields

$$P_{t+1} u'(C_t^Y) = \beta R_t P_t. \tag{A.9}$$

Thus the real rate r_t satisfies $r_t = R_t P_t / P_{t+1} = u'(C_t^Y)/\beta$.

Optimal production by entrepreneurs. Given date-t aggregate output Y_t, entrepreneur $i \in [0,1]$ posts the price P_i that solves

$$\max_{P_i} P_i Y_t^i - \frac{W_t Y_t^i}{\alpha}, \tag{A.10}$$

where $Y_t^i = Y_t(P_t/P_i)^{\epsilon}$ from (A.3). The first-order condition and that $P_i = P_t$ in equilibrium yields the real wage

$$\frac{W_t}{P_t} \equiv w = \alpha(1-\mu), \tag{A.11}$$

where

$$\mu = \frac{1}{\epsilon} \tag{A.12}$$

is entrepreneurs' mark-up—real profit per unit of output. Together with (A.8) and (A.9), this yields an equilibrium output Y_t and labor supply L_t

$$Y_t = \alpha L_t = \frac{\alpha^2\beta(1-\mu)r_t}{\gamma} \tag{A.13}$$

and so entrepreneurs' real profit is

$$\mu Y_t = \frac{\alpha^2\beta\mu(1-\mu)r_t}{\gamma} - f = \delta r_t - f \tag{A.14}$$

and households save

$$B_t = W_t L_t - C_t^Y = P_t\left(\frac{\alpha^2\beta(1-\mu)^2 r_t}{\gamma} - (u')^{-1}(\beta r_t)\right) = P_t\left[\frac{\delta(1-\mu)r_t}{\mu} - \phi(\beta r_t)\right]. \tag{A.15}$$

with the notations δ and $\phi(.)$ introduced in the body of the paper.

Optimal investment by entrepreneurs. Date-t entrepreneurs choose the share a_t of the profit $\mu Y_t - f$ that they invest in their technology with return ρ, the total investment size I_t, and a real stake in the proceeds $R_{E,t}$ that solve

$$\max_{\{a_t, I_t, R_{E,t}\}} \{R_{E,t} + r_t(\mu Y_t - f - a_t)\} \tag{A.16}$$

s.t.

$$\rho I_t - R_{E,t} \geq r_t(I_t - a_t), \tag{A.17}$$

$$R_{E,t} \geq (1-\lambda)\rho I_t, \tag{A.18}$$

$$a_t \in [0, \mu Y_t - f], \tag{A.19}$$

where (A.17) is the participation constraint of the households and (A.18) the incentive-compatibility constraint of the entrepreneurs. The former constraint can be rewritten as $R_{E,t} - r_t a_t \leq (\rho - r_t) I_t$, implying that $I_t = a_t = R_{E,t} = 0$ if $\rho < r_t$. It also implies that if $\rho > r_t$, then entrepreneurs maximize I_t. Combining (A.18) and (A.17) yields $(r_t - \lambda\rho) I_t \leq r_t a_t$. Thus the program has no solution if $r_t \leq \lambda\rho$. Otherwise, $a_t = \mu Y_t - f$, $R_{E,t} = (1-\lambda)\rho I_t$, and

$$I_t = \frac{(\mu Y_t - f) r_t}{r_t - \lambda\rho}. \tag{A.20}$$

Finally, if $\rho = r_t$, then any $I_t \in [0, (\mu Y_t - f)/[\rho(1-\lambda)]]$ solves the program with any $R_{E,t} = \rho a_t \geq (1-\lambda)\rho I_t$.

Bond-market clearing. Bond-market clearing then yields the real rate r_t^*. The central bank has a zero-net supply of bonds. Entrepreneurs supply bonds worth their investment needs net of their net wealth, and households' demand is B_t/P_t. There are three possible outcomes for I_t depending on the position of r_t^* relative to ρ:

1. $I_t = 0$ and entrepreneurs lend their profits to households so that $B_t = -(\delta r_t^* - f) P_t$, implying $r_t^* = r^* = (\phi(\beta r^*) + f)\mu/\delta > \rho$.
2. $r_t^* = \rho$ and $I_t \in [0, I(\rho)]$ is such that $I_t = \delta\rho/\mu - \phi(\beta\rho) - f$.
3. $r_t^* \in (\rho\lambda, \rho)$, and the external funds $I_t - \mu Y_t + f$ raised by entrepreneurs are equal to the households' savings B_t/P_t, or

$$\frac{(\delta r^* - f) r^*}{r^* - \lambda\rho} - (\delta r^* - f) = \frac{\delta(1-\mu) r^*}{\mu} - \phi(\beta r^*) \tag{A.21}$$

or

$$\frac{\delta (r^*)^2 - \lambda\rho f}{r^* - \lambda\rho} = \frac{\delta r^*}{\mu} - \phi(\beta r^*). \tag{A.22}$$

Notice in particular that this equation admits at most one solution because the LHS has a slope $(\delta r(r - 2\lambda\rho) + f\lambda\rho)/(r - \lambda\rho)^2 < \delta$ from (5) whereas the RHS has a slope larger than $\delta/\mu > \delta$. Simple differentation shows that I_t first decreases then possibly increases w.r.t. the interest rate in this range.

Determination of inflation. The Fisher equation (A.9) combined with the Taylor rule (3) yields for all $t \geq 0$

$$r^* \Pi_{t+1} = R_t = r^M \Pi^M \left(\frac{\Pi_t}{\Pi^M} \right)^{1+\psi}, \tag{A.23}$$

or

$$\text{(A.24)} \qquad \frac{\Pi_{t+1}}{\Pi_t} = \frac{r^M}{r^*}\left(\frac{\Pi_t}{\Pi^M}\right)^{\psi}.$$

The only price path that satisfies this and does not lead to exploding inflation rates is such that $\Pi_t = \Pi^* \equiv \Pi^M(r^*/r^M)^{1/\psi}$ for all $t \geq 0$.

Step 2. Optimality of adjusting the price. The flexible-price equilibrium can be sustained if entrepreneur $i \in [0,1]$ born at date t finds it preferable to optimize the price of the intermediate good P_t^i rather than leave it unchanged at P_{t-1}^i and save f when other agents adjust their prices:

$$\text{(A.25)} \qquad \max_{P^i}\left\{Y\left(\frac{P_t}{P^i}\right)^{\epsilon}\left(P^i - \frac{W_t}{\alpha}\right)\right\} - fP_t \geq Y\left(\frac{P_t}{P_{t-1}^i}\right)^{\epsilon}\left(P_{t-1}^i - \frac{W_t}{\alpha}\right)^{+},$$

which can be rewritten after optimizing over P^i, and using $P_t/P_{t-1} = \Pi^*$, $W_t/P_t = w = \alpha(1-\mu)$, and $\alpha\beta w r^* = \gamma Y$

$$\text{(A.26)} \qquad f \leq \delta r^*\left[1 - \frac{(\Pi^*)^{\frac{1-\mu}{\mu}}}{\mu}[1-(1-\mu)\Pi^*]^{+}\right].$$

The right-hand side is strictly positive if and only if $\Pi^* \neq 1$, and increasing with respect to r^*, and with Π^* over $[1,+\infty)$. Since r^* and thus Π^* are decreasing in f, this implies that if $\Pi^M(r^{*,0}/r^M)^{1/\psi} > 1$, there exists a threshold $\bar{f} > 0$ such that there exists a flexible -price equilibrium if and only if $f \leq \bar{f}$.

Step 3. Comparative statics. The comparative statics w.r.t. to λ directly result from the RHS of (A.22) being increasing in r, independent of λ whereas the LHS increases with respect to λ and its graph crosses that of the RHS from above.

A.2 Proof of Proposition 2

For each $b \in (0, B]$, the date-t real rate r_t and the bubble size b_t are recursively defined by

$$\text{(A.27)} \qquad b_0 = b,$$

$$\text{(A.28)} \qquad b_{t+1} = r_t b_t,$$

$$\text{(A.29)} \qquad S(r_t) - I(r_t) = b_t.$$

Since the real rate r_t is for all t larger than r^* and increasing in b, so is inflation $\Pi_t = \Pi^M(r_t/r^M)^{1/\psi}$, and so condition (23) is satisfied in the presence of a bubble if it is so without.

The only impact of bubbles on households' decision making is that they face a higher interest rate, which increases their utility from the application of the envelope theorem to their program (A.4).

If the bubble reduces investment, as is always the case when $I(.)$ is decreasing, then it reduces entrepreneurs' utility $(\rho - r_t)I_t$ at each date since it also raises r_t. To construct an example in which, conversely, a bubble lifts entrepreneurs' utility at each date, suppose (for simplicity) that $f = 0$, and that $r^* = 2\lambda\rho$, that is, r^* corresponds to the minimum of $I(.)$. In this case, entrepreneurs' utility is higher in the presence of a constant-size bubble B and a unit interest rate if and only if:

$$\frac{(\rho - 1)}{1 - \lambda\rho} \geq 4\rho^2(1 - 2\lambda)\lambda, \tag{A.30}$$

which holds if λ is sufficiently small all else equal. λ can always be taken sufficiently small as only the product $\lambda\rho$ enters into the equilibrium characterization.

A.3 Proof of Proposition 3

There are three claims that are not established in the derivation of the equilibrium in the body of the paper.

Equations $\{$(29);(30); (33);(34)$\}$ *pin down a unique* (C^Y, L, I, w). Condition (30) yields C^Y. Condition (29) yields w as a functions of L. Injecting it in (33) and (34) yields in turn

$$\frac{\gamma L^2}{\beta} - \alpha\rho\lambda L - \phi(\beta\hat{r})(\hat{r} - \rho\lambda) = 0, \tag{A.31}$$

which has a unique positive solution in L, then w stems from (29) and I from (34).

Entrepreneurs are willing to accommodate demand. This is so if the real wage is such that $w \leq \alpha$, equivalently $\gamma L/(\beta\hat{r}) < \alpha$ from (29). This is true because the LHS of (A.31) is strictly positive for $L = \alpha\beta\hat{r}/\gamma$ since $\hat{r} > \underline{r}$.

Monetary policy amounts to selecting the mark-up in the economy. In the flexible-price equilibrium, the counterpart of equations $\{$(29);(30); (33);(34)$\}$, together with profit maximization yielding (15) ($w = \alpha(1 - \mu)$), fully characterizes the real block of the model (C^Y, L, I, w, r). In the fixed-price model, this latter equation is missing but the real rate $\hat{r}$ stems from monetary policy. Thus one can get the fixed-price outcome as the outcome of an economy without menu costs in which the markup is derived from the fixed-price equilibrium wage, or, $\mu = 1 - w/\alpha$.

A.4 Proof of Proposition 4

The right-hand inequality in (36) is simply (23) ensuring the existence of the flexible-price equilibrium. The fixed-price equilibrium can be sustained if entrepreneur $i \in [0,1]$ born at date t finds it preferable to leave the price of good i unchanged at $P^i_{t-1} = P_{-1}$ and save f to optimal pricing when other agents behave as described in Proposition 3. Formally, denoting respectively $\hat{Y}$, $\hat{W}$, and $\hat{w}$ the respective output, nominal and real wages in the fixed-price equilibrium, it must be that

$$\text{(A.32)} \qquad \max_{P^i} \left\{ \hat{Y} \left(\frac{P_{-1}}{P^i} \right)^{\epsilon} \left(P^i - \frac{\hat{W}}{\alpha} \right) \right\} - f P_{-1} \leq \hat{Y} \left(P_{-1} - \frac{\hat{W}}{\alpha} \right),$$

which can be rewritten after optimizing over P^i

$$\text{(A.33)} \qquad f \geq \hat{Y} \left[\mu \left[\frac{\alpha(1-\mu)}{\hat{w}} \right]^{\frac{1-\mu}{\mu}} + \frac{\hat{w}}{\alpha} - 1 \right].$$

Using $\hat{w} = \alpha(1 - \mu(\hat{r}))$ and $\hat{Y} = \alpha^2 \beta (1 - \mu(\hat{r}))\hat{r}/\gamma$ then yields the left-hand side of (36).

If the real parameters and r^M are such that (36) holds for some $\hat{r} \in (\underline{r}, \rho)$ and $\Pi^* > 0$, then Π^M and ψ are the unique solution to

$$\text{(A.34)} \qquad \Pi^* = \Pi^M \left(\frac{r^*}{r^M} \right)^{\frac{1}{\psi}},$$

$$\text{(A.35)} \qquad \hat{r} = \frac{r^M}{(\Pi^M)^{\psi}},$$

and so ψ is given by

$$\text{(A.36)} \qquad (\Pi^*)^{\psi} = \frac{r^*}{\hat{r}},$$

and Π^M by

$$\text{(A.37)} \qquad (\Pi^M)^{\psi} = \frac{r^M}{\hat{r}}.$$

Finally, take f such that $\delta r^* - f > 0$. For $\Pi^* > 1/(1-\mu)$, the right-hand side of (36) holds. Take $\hat{r}$ sufficiently close to r^* that $\mu(\hat{r})$ is sufficiently close to $\mu(r^*) = \mu$ and the left-hand side of (36) also holds because the leftmost term tends to 0 as $\hat{r} \to r^*$. One can then from above select Π^M and ψ to reach such $(\Pi^*, \hat{r})$.

A.5 Proof of Proposition 5

Stochastic equilibrium. Let $p \in (0,1)$. Consider a stochastic process $(\tilde{\Omega}_t)_{t \geq 0}$ such that $\Omega_0 = 1$. At each subsequent date $t \geq 1$, $\tilde{\Omega}_t$ remains equal to 1 with probability p, or snaps to 0 with probability $1-p$, in which case it stays equal to this value forever after. The realizations of $\tilde{\Omega}_t$ are public information.

Claim. If p is sufficiently large, there exists a sunspot equilibrium such that:

- As long as $\tilde{\Omega}_t = 1$, prices are rigid, the policy rate is $\hat{r}$, the real rate $\hat{r}[p + (1-p)/\Pi^*]$.
- At the stopping time τ such that $\tilde{\Omega}_\tau = 0$, prices becomes flexible, CPI inflation jumps to Π^* and then stays at this level forever, and the real rate becomes r^*.

Proof. At the date τ at which $\Omega_\tau = 0$, the economy can revert to the non-bubbly flexible price equilibrium as the situation is the same as that of the perfect-foresight model at date 0.

Consider now the stochastic phase before $\Omega_\tau = 0$. Entrepreneurs being risk neutral and workers being risk neutral when old, the expected interest rate drives their decisions as the deterministic one does in the perfect-foresight equilibrium. For p sufficiently large, the expected interest rate $\hat{r}[p + (1-p)/\Pi^*]$, reflecting that the economy reverts to the flexible-price equilibrium with probability $1-p$ next period, is sufficiently close to $\hat{r}$ that a fixed-price equilibrium can exist given this rate by continuity.

Equilibrium with infrequent adjustment. For $x \in [0,1]$, define (w^x, r^x, Y^x, Π^x) as the solution to

(A.38) $$\alpha\beta w^x r^x = \gamma Y^x,$$

(A.39) $$\frac{\left[\left(1 - \frac{w^x}{\alpha}\right) Y^x - xf\right] r^x}{r^x - \lambda\rho} = Y^x - \phi(\beta r^x) - xf,$$

(A.40) $$r^x = r^M \left(\frac{\Pi^x}{\Pi^M}\right)^\psi,$$

(A.41) $$\Pi^x = \left[1 - x + x\left[\frac{w^x \Pi^x}{\alpha(1-\mu)}\right]^{\frac{1-\mu}{\mu}}\right]^{\frac{\mu}{1-\mu}}$$

Together with a proper distribution of initial prices, these four equations correspond to a steady state in which a fraction x of firms adjust their prices at each date. The case $x = 0$ corresponds to the fixed-price equilibrium and $x = 1$ to the flexible price one. In particular, (A.41) computes inflation given that the fraction that adjusts optimally chooses a price P such that

$P/P_t = w^x/[\alpha(1-\mu)]$. Notice that this equation imposes that w^x be equal to $\alpha(1-\mu)$ when $x = 1$. These variables define indeed an equilibrium if each firm is indifferent between adjusting its price or not after $\lfloor 1/x \rfloor$ dates. This means that x must solve:

$$\text{(A.42)} \qquad \mu Y^x \left[\frac{\alpha(1-\mu)}{w^x}\right]^{\frac{1-\mu}{\mu}} - f = Y^x \left[\frac{\alpha(1-\mu)}{w^x}\left(\Pi^x\right)^{\frac{1}{x}}\right]^{\frac{1-\mu}{\mu}} \left[1-(1-\mu)\left(\Pi^x\right)^{\frac{1}{x}}\right]^+$$

The left-hand side is larger than the right-hand one for $x = 1$ by definition of the flexible-price equilibrium and smaller for $x = 0$ by definition of the fixed-price one and so there is at least one solution by continuity.

A.6 Proof of Proposition 6

Claim. The equation

$$\text{(A.43)} \qquad \frac{\delta(x)\hat{r}^2}{\hat{r}-\lambda\rho} = \frac{\delta(x)\hat{r}}{x} - \phi(\beta\hat{r}) - b,$$

where $\delta(x) = \alpha^2\beta x(1-x)/\gamma$, admits a unique solution x in $(0, 1/2]$ if $b \in [0, b^{max})$. This solution is decreasing in b. (A.43) admits no solution over $(0, 1/2]$ if $b \geq b^{max}$.

Proof. The function $\delta(x)$ is an increasing bijection over $(0, 1/2]$ tending to 0 at 0 whereas $\delta(x)/x$ is decreasing, tending to $\alpha^2\beta/\gamma$. That there exists a non-bubbly fixed price equilibrium means that (A.43) has a (unique) solution in $(0, 1/2]$ when $b = 0$ since $\mu(\hat{r}) \leq 1/2$. Thus it also has one for any $b \in [0, b^{max})$, closer to the origin as b increases.

This shows that the set of old bubbles b_0 for which the equilibrium conditions $\{$(38);(39);(40);(41);(42)$\}$ hold is exactly $[0, b^{max})$, and that the equilibrium is uniquely defined for each value of b_0. The real block of the economy is isomorphic at date t to that in the flexible model with a mark-up μ^{b_t} decreasing in b_t.

The comparative statics then are a straightforward consequence from the fact that the shadow markup μ^{b_t} is decreasing in the size of a bubble b_t. Output $\delta(\mu^{b_t})\hat{r}/\mu^{b_t}$ and wage $\alpha(1-\mu^{b_t})$ increase at every date in the size of the initial bubble b^0 because so do the date-t bubble and thus μ^{b_t} decreases. The higher wage implies that households are better off at each date from the envelope theorem. The capital share $\delta(\mu^{b_t})\hat{r}$ decreases whereas the interest rate and thus the leverage ratio both remain unchanged, implying that entrepreneurs are worse off and investment smaller.

A.7 Proof of Proposition 7

If at a given date young entrepreneurs issue a fraction ω of a total bubble b, the shadow markup μ such that the capital market clears solves:

$$\text{(A.44)} \qquad \frac{\alpha^2\beta\hat{r}^2\mu(1-\mu)}{\gamma(\hat{r}-\lambda\rho)} + \frac{\omega b\hat{r}}{\hat{r}-\lambda\rho} = \frac{\alpha^2\beta\hat{r}(1-\mu)}{\gamma} - \phi(\beta\hat{r}) - (1-\omega)b.$$

Using the market-clearing condition (A.44) to eliminate b, one obtains that holding ω fixed, investment (e.g., the RHS of(A.44)) varies with μ as does the function $-(1-\omega)\mu^2 - (2\omega-1)\mu$. Thus it decreases in μ if $\omega \geq 1/2$, meaning that investment increases with respect to b. Similarly, investment increases in μ if $\omega \leq (1-2\mu(\hat{r}))/[2(1-\mu(\hat{r}))]$ since $\mu \leq \mu(\hat{r})$. Otherwise investment increases then decreases in b holding ω fixed.

Finally, condition (51) stating that old bubbles must be refinanced together with $\omega_t > (1 - 2\mu(\hat{r}))/[2(1-\mu(\hat{r}))]$ at all t implies explosive bubbles (b_{t+1}/b_t bounded away from 1) if $\hat{r} > 1/[2(1-\mu(\hat{r}))]$, which cannot be.

A.8 Proof of Proposition 8

Point 1. In any perfect-foresight equilibrium, denoting Π_t and r_t the respective date-t equilibrium values of inflation and the real rate, the combination of the Euler equation and the Taylor rule yields

$$\text{(A.45)} \qquad r_t\frac{\Pi_{t+1}}{\Pi_t} = r^M\left(\frac{\Pi_t}{\Pi^M}\right)^{\psi}.$$

The presence of a bubble requires that $\prod_{t=0}^{T} r_t$ be bounded. Since Π_t is bounded as well, multiplying (A.45) between 0 and T term by term and raising to power $1/T$ yields the result as the left-hand side must be smaller than one for T sufficiently large.

Point 2. Let $p \in (0,1)$. Consider a stochastic process $(\tilde{\Omega}_t)_{t\geq 0}$ such that $\Omega_0 = 1$. At each subsequent date $t \geq 1$, $\tilde{\Omega}_t$ remains equal to 1 with probability p, or snaps to 0 with probability $1-p$, in which case it stays equal to this value forever after. The realizations of $\tilde{\Omega}_t$ are public information. Let us also construct a strictly positive sequence $(b_t)_{t\in\mathbb{N}}$ such that $b_{t+1} = \hat{r}[1+(1-p)/(p\Pi^*)]b_t$.

Claim. If p is sufficiently large and b_0 sufficiently small other things being equal, there exists a sunspot equilibrium such that:

- As long as $\tilde{\Omega}_t = 1$, prices are rigid and agents trade a monetary bubble with date-t value b_t. The policy rate is $\hat{r}$, the real rate $\hat{r}[p+(1-p)/\Pi^*]$.

- At the stopping time τ such that $\tilde{\Omega}_\tau = 0$, the bubble bursts, prices becomes flexible, and CPI inflation jumps to Π^* and then stays at this level forever, so that the policy rate becomes $r\Pi^*$ and the real rate becomes r.

Proof. At the date τ at which $\Omega_\tau = 0$, the economy can revert to the non-bubbly flexible price equilibrium as the situation is the same as that of the perfect-foresight model at date 0.

Consider now the stochastic phase before $\Omega_\tau = 0$. Entrepreneurs being risk neutral and workers being risk neutral when old, the expected interest rate drives their decisions as the deterministic one does in the perfect-foresight equilibrium. For p sufficiently large, the expected interest rate $\hat{r}[p + (1-p)/\Pi^*]$, reflecting that the economy reverts to the flexible-price equilibrium with probability $1-p$ next period, sufficiently close too $\hat{r}$ for p sufficiently large. The bubble must earn this rate on average but may burst next date with probability p, implying that it grows at the rate $\hat{r}[1 + (1-p)/(p\Pi^*)]$ as long as $\tilde{\Omega}_t = 1$. This rate is smaller than 1 provided p is sufficiently large. This ensures that for such p sufficiently large and b_0 sufficiently small that the markup in the presence of the bubble is sufficiently close to $\mu(\hat{r})$, (36) holds during the fixed-price phase of the equilibrium.

A.9 Proof of Proposition 9

Suppose that f is such that the rigid-price equilibrium in Proposition 3 cannot be sustained at the zero lower bound η, or, from condition (36), that

$$\eta \left[\left(\frac{1-\mu}{1-\mu(\eta)} \right)^{\frac{1-2\mu}{\mu}} - \frac{\mu(\eta)(1-\mu(\eta))}{\mu(1-\mu)} \right] \geq \frac{f}{\delta} \tag{A.46}$$

The left-hand side is strictly increasing in $\mu(\eta)$ if $\mu(\eta) > \mu$. Thus, a monetary bubble, by pushing down the value of $\mu(\eta)$, can ensure that (A.46) no longer holds.