Equilibrium Subprime Lending

IGOR MAKAROV and GUILLAUME PLANTIN*

ABSTRACT

This paper develops an equilibrium model of a subprime mortgage market. Our goal is to offer a benchmark with which the recent subprime boom and bust can be compared. The model is tractable and delivers plausible orders of magnitude for borrowing capacities, as well as default and trading intensities. We offer simple explanations for several phenomena in the subprime market, such as the prevalence of teaser rates and the clustering of defaults. In our model, both nondiversifiable and diversifiable income risks reduce debt capacities. Thus, debt capacities need not be higher when a larger fraction of income risk is diversifiable.

One of the main goals of the financial system is to make credit widely available to potential borrowers. The rapid growth of the U.S. subprime mortgage market was viewed as an important advance in this direction because banks provided credit to a group of individuals that previously were not considered creditworthy. Subprime lending practices came under close scrutiny, however, after a surge in delinquencies sparked a severe financial crisis.[1] The general belief is that banks supplied more credit than individuals could afford to repay. This leads to two important questions: what is the borrowing capacity of the subprime population, and what are the resulting default rates when borrowers exhaust this capacity? To investigate these questions, this paper develops an equilibrium model of a subprime mortgage market.

Several features of our model are worth noting. Consistent with evidence from the subprime population,[2] we assume that households' preferences and endowments are such that in equilibrium, their demand for housing is constrained by the amount that they can borrow from banks. Further, we assume that banks compete to offer households the largest possible mortgages. The model incorporates two contracting frictions, which affect mortgage size. First, households privately observe their income, and thus banks cannot verify households' reports, which is considered a characteristic of subprime borrowers. Second, households cannot commit to a contract and are free to terminate it at any time. By including this assumption, we aim to capture the fact that subprime borrowers have little to lose apart from their home when taking on a mortgage. These two frictions force banks to commit to foreclosures in order to incentivize households to repay their loans. This foreclosure process, however, is costly and takes a fraction of the house market value. Households' aggregate borrowing capacity drives the aggregate demand for homes. At the same time, market-clearing home prices affect aggregate borrowing capacity. Thus, household borrowing capacities and home prices are jointly determined in equilibrium.

The model is analytically tractable and suggests plausible orders of magnitude for loan-to-income ratios, as well as default and trading intensities. In addition, it offers several qualitative and quantitative insights.

First, the degree of income comovement among households is an important determinant of borrowing capacity. Both nondiversifiable and diversifiable income risks reduce debt capacities, although through quite distinct channels. As a result, the degree of income comovement among households has an ambiguous effect on borrowing capacities and home prices.

When income is correlated across households, foreclosures are more likely to take place when aggregate income is low. This implies that foreclosed homes can only be resold at lower prices. This reduces the collateral value of homes and leads to lower

borrowing capacities and home prices. This effect vanishes when income is uncorrelated across households because foreclosure on a particular household is uncorrelated with home prices.

However, while uncorrelated income across households implies that buyers are available to pay higher prices on foreclosed homes, it also implies that those households with positive income shocks are now more likely to terminate their existing contracts and take on a larger mortgage in order to move to a larger house. Thus, banks cannot maintain a long-lived relationship with their most solvent borrowers, which results in lower mortgage capacity. The same set of dynamics does not unfold when household income is correlated because a household that experiences a positive income shock would compete in mortgage and housing markets with other households that have similarly high income realizations. This makes terminating a current mortgage less valuable to the household and therefore less costly to its bank.

Second, while most of our results obtain under the assumption that banks use debt contracts with constant repayments, we also consider debt contracts with nonconstant repayments. We demonstrate that contracts with initial "teaser rates" that gradually increase over time lead to higher household borrowing capacity for a wide range of parameters. These types of contracts were very common in subprime mortgages and are often linked to the roots of the crisis. Indeed, banks have been portrayed as villains who misled naive households into taking too much debt using teaser rates. Setting aside any normative issues, we show that these contracts maximize household borrowing capacities. The intuition is quite simple: even though the income process has no drift and there is no inflation risk in our model, conditioning on the household continuing to make payments, its expected income increases over time. This lowers the conditional probability of default, which in turn makes it optimal for banks to ask for increasing payments over time.

Finally, we also solve for the dynamics of home prices along the path to the steady-state after an initial exogenous loan supply shock, which can be, for example, an exogenous shift in securitization practice. We find that defaults peak three to four years after the initial loan supply shock and that banks offer riskier loans at the initial date because they anticipate a home price correction, and thus prefer to foreclose sooner rather than later. We show that anticipating a 15% home price decrease along the transition path, banks initially grants loans that are 30% lower than in the steady-state. Thus, loan size is very sensitive to expected future variations in home prices.

This paper brings together two strands of literature — the literature on the microeconomics of mortgages and the literature on endogenous incomplete markets. Mayer, Piskorski, and Tchistyi (2008) and Piskorski and Tchistyi (2008) also study optimal

mortgage design in the presence of ex post informational asymmetry. Our papers are complementary. They solve for the optimal recursive contract between a lender and a long-sighted borrower in an exogenous environment. We focus on the aggregate implications of contracting frictions between many pairs of lenders and short-sighted borrowers.

Stein (1995) and Ortalo-Magné and Rady (2006) study the impact of credit constraints on prices and trading volume in housing markets. Both papers assume exogenous credit constraints, and write a detailed model of households' decision-making under such constraints. Unlike these contributions, we study only the highly constrained subprime segment of the population, but we derive credit constraints as endogenous consequences of primitive frictions.

A second body of work studies how inability to commit to a contract prevents agents from optimally sharing risk.[3] Closest to our approach, Krueger and Uhlig (2006) introduce the idea that households' outside options after terminating a contract are competitively supplied by a financial sector, which is an important ingredient of our model. We have a more applied focus than these papers, but share the broader goal of characterizing the equilibrium interaction between individual contracting problems and asset prices.

The paper is organized as follows. Section I. outlines the model. Section II. solves for the steady-state in the particular case in which banks use standard debt contracts with constant repayments. Section III. studies out-of-steady-state dynamics. Section IV. discusses optimal contracts. Section V. studies the effect of aggregate income shocks on this economy. Section VI. concludes. Technical proofs are relegated to the Appendix.

I.   Baseline Model

Time is continuous and is indexed by $t \in [0, +\infty)$. There is a single perishable consumption good that serves as the numéraire. There is a unit mass of housing units. There are two types of agents, a unit mass of households and several non-atomistic banks.

A. Households

Households derive utility from consumption c_t and occupied housing units q_t. Each household ranks bundles $(c_t, q_t)_{t \geq 0}$ according to the criterion

$$E \left(R \int_0^{+\infty} e^{-Rt} u \left(c_t, q_t \right) dt \right),$$

where $R > 0$ and u is increasing, concave, and differentiable. Households can only consume and occupy nonnegative quantities. We interpret c_t as the fraction of consumption that exceeds a minimum subsistence level, which we normalize to zero. Let ψ denote the lower bound of the marginal preference for housing over consumption:

$$\psi = \min_{c \geq 0, q \geq 0} \left\{ \frac{\partial u}{\partial q} \bigg/ \frac{\partial u}{\partial c} \right\}.$$

We restrict the analysis to the case in which the discount rate R and ψ are very large.[4] We will show that in this case, binding borrowing constraints determine mortgage and housing demand because households are always willing to trade current consumption for more housing in equilibrium.

For each household $j \in [0, 1]$, there exists a Poisson process $(N_{j,t})_{t \geq 0}$ with intensity $\delta > 0$ such that at each arrival time, household j vacates its current home, and re-enters the housing market for unmodelled reasons. These events capture trades in the housing market that are not primarily driven by the evolution of the real estate market, but rather by occupational changes, changes in household size, etc.[5] In the remainder of the paper, we refer to these events as exogenous termination dates, or ET dates. Let $\aleph_j$ be the set of ET dates for household $j \in [0, 1]$. Each household $j \in [0, 1]$ is endowed with an income stream that consists of the subsistence level plus a disposable fraction $(I_{j,t})_{t \geq 0}$ such that

$$\begin{cases} \forall t \in \aleph_j, I_{j,t} = 1 \\ \frac{dI_{j,t}}{I_{j,t}} = \sigma dW_{j,t}, \text{ for } t \notin \aleph_j \end{cases}, \tag{1}$$

where $W_{j,t}$ is a standard Brownian motion and $\sigma > 0$. Thus, households' income follows a geometric Brownian motion, which is reset at each ET date to one — the unconditional average income of all households in the economy. If income was not reset at ET dates, the average income of the households that move for exogenous reasons at a given date would still be equal to one because ET dates are independent of income risk. Re-setting household income processes ensures that the cross-section of incomes has a stationary distribution. The assumption that income reset dates and ET dates are generated by the same Poisson process is made for analytical convenience. Re-setting

income at other dates than the ET dates would complicate the analysis without adding any insights. All stochastic processes $\left((W_{j,t})_{t \geq 0}, (N_{j,t})_{t \geq 0} \right)$ are assumed to be pairwise independent. Thus, there is no aggregate uncertainty in this baseline model.

B. Banks

Banks care only about consumption. They are infinitely lived and risk-neutral, and they discount the future at the rate $r > 0$. Banks are not financially constrained: we assume that their aggregate endowment is always larger than households' aggregate debt capacity. Our goal is to determine the potential mortgage capacity of subprime borrowers in the presence of an efficient financial system, and thus we abstract from banks' financial constraints.

Banks own an eviction technology. At any date, a bank can terminate a relationship with a household by evicting the household and sell its units in the market for vacant homes. However, eviction comes at a cost equal to a fraction λ of the home market value, where $\lambda \in (0, 1]$. This captures the value loss implied by foreclosures.

C. Market for Vacant Homes

Banks and households are home-price takers. Vacant housing units are perfectly divisible. However, to modify its home size from q to $q' \neq q$, a household needs to move into q' new units, which entails transaction costs: if a household moves at date t, then for all $s \in [t, t + \Delta t]$ its instantaneous utility is only

$$u \left(c_{t+s}, \chi q_{t+s} \right),$$

where $\Delta t > 0$ is small and $\chi \in [0, 1)$. That is, only a fraction χ of its new housing units enters its preferences. This captures all the costs associated with search in the housing and mortgage markets, moving, home remodelling, etc. This temporary utility loss vanishes after $t + \Delta t$. In what follows, therefore, we refer to χ as the relocation cost parameter, and to its transformation $\chi^{-1} - 1$ as the relocation costs. These costs imply that households move only at discrete dates.

D. Contracting Frictions

Banks enter into individual financing contracts with households. Two contracting frictions affect mortgage size. First, households privately observe the realization of their income paths and consume secretly. Thus, banks cannot verify households reports,

which is considered a characteristic of subprime borrowers. Other than households' income realizations, everything else in this economy is publicly observable.

Second, households cannot commit to a contract and are free to terminate it at any time. By including this assumption, we aim to capture the fact that subprime borrowers have little to lose apart from their home when taking on a mortgage. Extensions could endow lenders with recourse possibilities, and study how it affects debt capacities. Banks can fully commit to a contract with a household.

E.   Equilibrium Concept

At each date, banks simultaneously post contract menus. If several banks make identical competitive offers, they obtain equal market shares. We solve for a steady-state in which there is no aggregate uncertainty. More precisely, a steady-state is characterized by a contract menu, a price per housing unit, and a distribution of housing units across households such that at each date three conditions are satisfied. First, a bank cannot make a strictly positive profit by offering a different menu of contracts. Second, the market for vacant housing units clears. Finally, each household has an optimal housing and consumption allocation given its current contractual obligations and income, the available contract menus, and the housing price.

We study this model as follows. First, in Section II., we restrict the space of contracts offered by banks to debt contracts with constant repayment schedules. This particular case is very tractable and allows us to obtain most of our results in closed form. Section III. studies price and default dynamics along paths to the steady-state. Section IV. discusses optimal contracts.

II.   Steady-State With Fixed Repayment Contracts

A.   Equilibrium

Suppose for now that banks can offer only standard debt contracts with fixed repayment schedules. In this contract, each bank quotes two numbers, a loan-to-income ratio $L \geq 0$ and a repayment ratio $\kappa \in [0, 1]$. A household that makes an initial payment of i receives a loan equal to Li and is asked to repay κi per unit of time from then on. If the household fails to make a payment, the bank commits to evict the household, and the contract is terminated. The next lemma describes the behavior of impatient and constrained households facing such contracts.

6

LEMMA 1: *If ψ and R are sufficiently large, then each household takes the largest possible loan at the outset of a contract and invests the loan entirely in housing. The household meets the repayment schedule until the first of the following three events occurs. 1. An exogenous termination date is realized and the household terminates the contract for exogenous reasons. 2. Household income falls below the repayment value and the household is evicted. 3. Household income increases sufficiently and the household voluntarily moves out and buys a larger home. As ψ, $R \to \infty$, the income threshold at which the household voluntarily moves out tends to $\chi^{-1} I_t$, where I_t is the household's income at the beginning of the contract.*

Proof: See the Appendix.

Lemma 1 states that sufficiently impatient and liquidity-constrained households demand the largest possible mortgage at the outset of a contract, and stay in their home until they have to move for exogenous reasons, they get evicted because they cannot repay their debt, or they choose to move to a larger house.

The remainder of this section characterizes the steady-state in the case in which ψ and R are sufficiently large that Lemma 1 applies and households move voluntarily when their income grows by χ^{-1}. Let

$$\alpha = -\frac{1}{2} + \sqrt{\frac{2(r+\delta)}{\sigma^2} + \frac{1}{4}}, \; \rho = 1 - \lambda\left(1 + \frac{\delta}{r}\right), \; \omega = \sqrt{\frac{2\delta}{\sigma^2} + \frac{1}{4}}.$$

The following proposition shows that we can analytically solve for the steady-state up to the repayment ratio κ, which is the solution to an algebraic equation.

PROPOSITION 1: *Suppose ψ and R are sufficiently large that households behave as described in Lemma 1. If banks quote a repayment ratio $\kappa \in [0, 1]$, then the average income of households that initiate a new contract at a given date is given by*

$$I = \frac{1 - p_\kappa - p_\chi}{1 - \kappa p_\kappa - \frac{p_\chi}{\chi}}, \tag{2}$$

where

$$p_\chi = \frac{\sqrt{\chi}\left(\kappa^\omega - \kappa^{-\omega}\right)}{\left((\kappa\chi)^\omega - (\kappa\chi)^{-\omega}\right)}, \tag{3}$$

$$p_\kappa = \frac{\left(\chi^\omega - \chi^{-\omega}\right)}{\sqrt{\kappa}\left((\kappa\chi)^\omega - (\kappa\chi)^{-\omega}\right)}. \tag{4}$$

Households arrive in the market with intensity $\delta/(1 - p_\kappa - p_\chi)$. This arrival intensity is the sum of the arrival intensities of households that i) unwittingly default, ii) volun-

tarily terminate their contract in order to relocate, and iii) move for exogenous reasons at an ET date. These three intensities are respectively equal to $\delta p_\kappa / (1 - p_\kappa - p_\chi)$, $\delta p_\chi / (1 - p_\kappa - p_\chi)$, and δ. Further, for a given repayment ratio κ, the loan-to-income ratio L is

$$L = l\left(\kappa\right) = \frac{\kappa}{r} \times \left[1 - \frac{1-\rho}{\frac{\kappa^{\alpha+1}(1-\chi^\alpha)+\kappa^{-\alpha}(\chi^{-\alpha-1}-1)}{(\chi^{-\alpha-1}-\chi^\alpha)} - \rho}\right]. \tag{5}$$

The equilibrium repayment ratio is $\arg\max l(\kappa)$. The price of a housing unit P is equal to LI.

Proof: A contract initiated by household j at date t is terminated at date $t + T^{j,t}$, where

$$T^{j,t} = \min\left(T_\delta^{j,t}, T_\kappa^{j,t}, T_\chi^{j,t}\right). \tag{6}$$

The three stopping times $T_\delta^{j,t}, T_\kappa^{j,t}$, and $T_\chi^{j,t}$ are defined as follows. First,

$$T_\delta^{j,t} = \min \aleph_j \cap (t, +\infty)$$

is the time passed until the first ET date after the contract is initiated. Second,

$$
\begin{aligned}
T_\kappa^{j,t} &= \min\left\{\tau \geq 0 : I_{j,t+\tau} = \kappa I_{j,t}\right\} \\
&= \min\left\{\tau \geq 0 : W_{t+\tau}^j - W_t^j = \frac{\ln\kappa}{\sigma} + \frac{\sigma\tau}{2}\right\}
\end{aligned}
$$

is the time passed until the income of household j falls below the value $\kappa I_{j,t}$ for the first time after t. Finally,

$$
\begin{aligned}
T_\chi^{j,t} &= \min\left\{\tau \geq 0 : \chi I_{j,t+\tau} = I_{j,t}\right\} \\
&= \min\left\{\tau \geq 0 : W_{t+\tau}^j - W_t^j = -\frac{\ln\chi}{\sigma} + \frac{\sigma\tau}{2}\right\}
\end{aligned}
$$

is the time passed until the income of household j reaches the value $\chi^{-1}I_{j,t}$ for the first time after t. Equation (6) states that a contract is terminated for one of the following reasons: i) an ET date occurs, ii) the household cannot meet a repayment, iii) the household voluntarily terminates the contract and vacates its home to acquire a larger one. Notice that the distributions of $T_\delta^{j,t}, T_\kappa^{j,t}$, and $T_\chi^{j,t}$ do not depend on j, t. Therefore, we omit the superscripts j, t for notational simplicity. Let

$$
\begin{aligned}
p_\chi &= \operatorname{Prob}\left(T_\chi < T_\kappa; T_\chi < T_\delta\right), \\
p_\kappa &= \operatorname{Prob}\left(T_\kappa < T_\chi; T_\kappa < T_\delta\right).
\end{aligned}
$$

8

The proofs of (3) and (4) are in the Appendix.

We now compute I, the average income of households that initiate a new contract at a given date. Consider a household in the market at date t with income $I_{j,t}$. The household's next arrival to the market will be because of an ET date, with an expected income equal to one, because of an eviction, with an income of $\kappa I_{j,t}$, or because of a decision to upgrade its home after its income reaches $\chi^{-1} I_{j,t}$. Averaging over all households in the market gives

$$I = 1 - p_\kappa - p_\chi + p_\kappa \kappa I + p_\chi \chi^{-1} I,$$

which yields (2).

Let $M dt$ denote the steady-state measure of households in the market between t and $t + dt$. A fraction $1 - p_\kappa - p_\chi$ of these households trade because of the realization of an ET date. Since such dates occur with intensity δ,

$$(1 - p_\kappa - p_\chi) M dt = \delta dt.$$

Therefore, the arrival intensity of households in the market M is $\delta / (1 - p_\kappa - p_\chi)$, and the respective arrival intensities of households that default and voluntarily relocate are $\delta p_\kappa / (1 - p_\kappa - p_\chi)$ and $\delta p_\chi / (1 - p_\kappa - p_\chi)$.

Let us finally compute L and P. Let S denote the quantity of vacant units per household in the market. If household j purchases $q_{j,t}$ units at date t, then it must the case that it receives a loan that satisfies

$$L I_{j,t} = E_t \left(\int_0^T e^{-rs} \kappa I_{j,t} ds + P \times q_{j,t} e^{-rT} (1 - \lambda \mathbf{1}_{\{T_\kappa < T_\kappa, T_\kappa < T_\delta\}}) \right).$$

The first term on the right-hand side is the present value from future repayments, and the second term is the present value from reselling the vacant home at the random date at which the contract ends. Integrating over the households in the market at date t yields

$$L I = \frac{\kappa I}{r} \left(1 - E e^{-rT} \right) + P S \times E \left[e^{-rT} \left(1 - \lambda \mathbf{1}_{\{T_\kappa < T_\kappa, T_\kappa < T_\delta\}} \right) \right].$$

Market clearing implies $PS = LI$, so that

$$L = \frac{\kappa}{r} \times \frac{1 - E e^{-rT}}{1 - E e^{-rT} + \lambda E \left(e^{-rT} \mathbf{1}_{\{T_\kappa < T_\chi, T_\kappa < T_\delta\}} \right)}. \tag{7}$$

The computations of $E e^{-rT}$ and $E \left(e^{-rT_\kappa} \mathbf{1}_{\{T_\kappa < T_\chi, T_\kappa < T_\delta\}} \right)$ that lead to expression (5) are relegated to the Appendix. The equilibrium repayment ratio is the one that

9

maximizes the loan-to-income ratio.

Finally, we have $S = 1$. Default and relocation times are independent of household home size. This implies that the sample of houses in the market at any point in time is a random sample of the economy-wide distribution of housing sizes. In particular, the quantity of housing per household in the market is equal to the quantity of housing per household in the economy, that is, $S = 1$. ∎

Notice that as $\lambda \to 0$, the loan-to-income ratio L tends to the level $1/r$ that would prevail without contracting frictions. Absent eviction costs, committing to eviction upon default comes at no cost.

A.1. Borrowing Capacity and Housing Price

The primitive parameters of the model δ, r, χ, λ, and σ determine only the loan-to-income ratio L and the average income in the market I. Our assumption of an inelastic unit supply then yields the home price $P = LI$. Alternatively, if we set the home price to an exogenous level P^* and let home supply adjust to the market-clearing level, then L and I would not change, and the home supply would be LI/P^*. Households' steady-state debt capacities depend only on how much a household invests in the housing market. As long as the home price is constant, this amount does not depend on a particular price level. This happens because households in our model are liquidity constrained and therefore have inelastic demand for housing. Thus, aggregate demand LI does not depend on the decomposition of supply into price and quantity.

A.2. Social Optimum

Ex ante efficient allocations would allow households to have constant home size over time and to frontload their consumption by borrowing from banks. Contracting frictions take the equilibrium away from this first-best. A full-fledged analysis of the welfare implications of these contracting frictions would require that we specify the opportunity sets and preferences of those agents who set up banks and those who supply housing. In this paper, we do not pursue this route but focus more simply on the determinants and magnitudes of equilibrium borrowing capacities L and home prices P.

B.   Orders of Magnitude

This section presents a simple calibration exercise in which we show that the model generates empirically plausible orders of magnitude for the steady-state aggregate quantities. We set the (real) rate r to 2%. We set the volatility of households' income σ to 25%, which is consistent with Dynan, Elmendorf, and Sichel (2008), who estimate

the volatility of pre-tax total household income at 23% to 25% for the 2002 to 2004 period in the PSID data set.[6] Based on a recent empirical study by Campbell, Giglio, and Pathak (2011), we set foreclosure costs λ to 30%. This leaves two free parameters: the arrival intensity of ET dates δ and the relocation cost parameter χ. As shown in the Internet Appendix, the cross-section of household income in our model has a double Pareto distribution that depends only on σ and δ.[7] With $\sigma = 25\%$, a maximum likelihood estimation of δ using the PSID sample yields $\delta = 6.25\%$. The relocation cost parameter χ affects households' moving decision. We set χ to 0.625 to deliver an effective mortgage duration of five years on average, which is plausible for the subprime segment.

These parameter values yield a steady-state repayment ratio κ of 42%, an arrival intensity of households to the market $\delta / (1 - p_\kappa - p_\chi)$ of 20%, a default intensity of 6.6%, and an average income of households conditional on trading in the housing market I of 1.1.

These estimates compare to their recent empirical counterparts as follows. Fabozzi (2006) reports average front-end debt-to-income ratios of 40% for subprime borrowers, and back-end ratios averaging above 50%.[8] We predict a fairly similar steady-state repayment ratio of 42%. In comparison, repayment ratios for the prime segment are typically below 30%. The trading intensity of 20% corresponds to an effective mortgage lifespan of five years, below the average effective duration of seven years over prime mortgages.

Assessing the plausibility of our default intensity is less straightforward because several definitions of mortgage default are used in practice. The broadest definition is delinquency, where payment delays exceed 60 days, and the narrowest definition is foreclosure. U.S. data yield annual delinquency rates on subprime mortgages varying between 8% and 17% between 1998 and 2007, while annual subprime foreclosure rates range from 2% to 9% over the same period.[9] Annual delinquency and foreclosure rates never exceed, respectively, 4% and 1.5% for prime mortgages over the same 10 years. Our model, in which delinquency and foreclosure coincide, predicts a default intensity of 6.6% that seems broadly in line with average subprime foreclosure rates observed over the 1998 to 2007 period. We consider a theoretical steady-state while the data corresponds to a subprime boom and bust. Thus, our default intensity of 6% to 7% gives a sense of what one could expect from a stable subprime market.

Finally, our model yields predictions on the distribution of the incomes of households active in the market at a given date relative to the unconditional income distribution. Predictions such as $I = 1.1$ are novel to our knowledge and could be tested in principle.

C. Comparative Statics

Proposition 1 maps the five primitive parameters σ, δ, λ, r, and χ into a full characterization of the steady-state. The comparative statics properties of the steady-state with respect to χ are particularly instructive. Recall that we interpret $\chi^{-1} - 1$ as relocation costs for households in mortgage and housing markets. Arguably, these costs have dramatically declined for subprime borrowers during the subprime boom. The dramatic growth in the size of the subprime segment must have made matching between homes, households, and subprime brokers much swifter. It is therefore interesting to study the impact of an increase in χ on the equilibrium variables in our model. Figure 1 shows equilibrium values of loan-to-income and repayment ratios, price and default rates as a function of χ.

[Figure 1 About Here]

From Figure 1 we can see that the equilibrium loan-to-income ratio decreases with respect to χ. This can be explained as follows. If a household terminates the contract and vacates its place to climb up the property ladder after a positive income shock, it destroys value for the bank because the household has become more solvent than it was when the loan was originated. The relocation costs $\chi^{-1} - 1$ act as a commitment device. The higher the χ, the smaller the relocation costs. Therefore, it is more likely that a household decides to leave its current home and buy a larger one after a positive income shock. Banks rationally anticipate this behavior and lower the initial loan size.

As the loan-to-income ratio decreases, so does the repayment ratio κ, which implies a lower probability of default of a given loan.[10] It would be tempting to conclude that both home prices and defaults decrease when relocation costs are smaller and their commitment role is thus reduced. Figure 1 shows that this is actually not the case. Even though each loan taken individually is smaller and less risky, households enter new contracts and thus maximize their debt capacity more often. As a result, aggregate lending increases, and the cross-section of borrowers becomes riskier. Higher aggregate lending leads, in turn, to higher equilibrium home prices. In other words, a decrease in relocation costs has a negative effect on each particular lender–borrower relationship because it makes the borrower's commitment problem more severe. On the other hand, reduced commitment power induces households to reveal and pay out their positive income shocks more frequently. This positive effect on information production is the dominant one on aggregate lending.

Overall, a decrease in relocation costs leads to a steady-state with higher aggregate lending and higher default rates despite safer and smaller individual loans. In practice, however, the change in defaults would occur only with a lag since it takes time for

households to hit the default boundary of a fresh mortgage. Failure to correctly antic-ipate this lagged change in defaults may have contributed to the mispricing of many subprime-backed securities.

III. Transition Dynamics

The steady-state analysis gives a sense of what should prevail in a stable subprime market. The rapid development of the subprime segment in the 2000s, however, is better thought of as a transition from an initial situation with very limited subprime borrowing to a new steady-state. In particular, many observers argue that changes in the securitization market led to an unanticipated credit supply shock (see, for example, Mian and Sufi (2009)). The goal of this section is to understand the out-of-steady-state dynamics resulting from such a shock.

A positive credit supply shock has two features: an increase in aggregate lending, and an increase in the number of new loans originated at a particular point in time. Our approach here is to capture these features in the simplest possible out-of-steady-state analysis.

Suppose that at the outset $t = 0$, all homes are vacant and no household is indebted. Let I_t denote the average income of households that are in the market at date t, P_t the date-t price of one housing unit, and S_t the date-t home supply in the market per household in the market. We also denote the respective loan-to-income ratio and repayment ratio that banks offer at date t as L_t and κ_t. At each date t, banks set L_t and κ_t to maximize the loan size available to households at date t given a projected price path P_{t+s} for $s \geq 0$. For simplicity, we set $\chi = 0$, so that loan termination occurs only at default dates and ET dates. Our goal is to determine the evolution of aggregate quantities along the path to the steady-date described in Proposition 1.

We start our computation assuming that at all dates t, banks give the steady-state contract described in Proposition 1. For this contract, we compute arrival intensities of households to the market, the average income path I_t, and the home price path P_t. Then, given the home price path P_t, we recompute κ_t and L_t to ensure that banks maximize the loan size at each date t. This leads to new arrival intensities of households to the market, a new average income path I_t, and a new home price path P_t. We iterate this procedure until convergence of all quantities. The Internet Appendix provides a detailed discussion of the solution technique.

Figure 2, Panels (A) and (B) depict the evolutions of the home price and default

intensity along the transition path.

[Figure 2 About Here]

In both graphs, the dashed line represents the path that would prevail if banks were naively offering the steady-state contract (κ, L) described in Proposition 1 from date 0 on. The solid line corresponds to the situation in which banks optimize (κ_t, L_t) at each date. In either case, there is a surge in default intensity three to four years after the opening of the mortgage market. The foreclosure rate exceeds 8% at this point before reverting back slowly to its steady-state level. Accordingly, home price decreases. The intuition is simple. Average income in the market starts out at date 0 at a maximal level. Then the fraction of households that trade because of an eviction increases because realized defaults on date-0 mortgages tend to cluster around the date at which the density of default times reaches its maximum. This lowers average income in the market and thus the price.

Figure 2 Panel (C) depicts the evolution of the repayment ratio κ_t. Notice first that the variation of κ_t is quite small. Yet, banks tend to become more conservative over time by setting decreasing repayment ratios. The intuition is that banks expect a sharp negative rate of return on housing at date 0. Thus, they are more willing to raise the first repayments, even if it triggers more early defaults. Postponing default with lower repayments is unappealing because doing so comes at the cost of larger reductions in housing prices once default occurs. Notice that these slightly higher out-of-steady-state repayment ratios significantly increase maximal default intensity from 7.5% to almost 8.5%: The peak of default intensities is very sensitive to the repayment ratios.

It is interesting to compare these orders of magnitude with facts from the subprime crisis. That the foreclosure rate exceeds 8% at its peak, which is reached about four years after the beginning of the boom, seems broadly in line with the facts over the 2003 to 2007 period. Figure 2, Panel (D) shows the evolution of the loan-to-income ratio L_t. We can see that anticipation of the 15% home price decrease in Panel (A) leads to the 30% decrease in loan size. Thus, loan size is very sensitive to the expected future variations in home prices. This is consistent with recent findings by Mian, Rao, and Sufi (2012), who find a strong correlation between home price fluctuations and household debt and debt-to-income over the 2002 to 2006 period, particularly in areas populated by the most constrained borrowers.

Finally, we would like to highlight the role of the simplifying assumption $\chi = 0$. Setting χ to zero makes the steady-state average income of households in the market be below the average income of households in the market at date 0. This implies that

the amount spent on housing is higher at the time of the shock and then decreases towards the steady-state. Higher values of χ can result in the steady-state average income of households in the market being above the average income of households in the market at date 0. We conjecture that this is likely to lead to an increase in home prices. Solving for the transition path in this case, however, is a challenging numerical problem that we leave for future research.

IV.   Optimal Repayment Schedules

So far we have restricted the analysis to the tractable case in which banks use only constant repayment schedules. This section discusses optimal repayment schedules. By optimal, we mean schedules designed by banks so as to maximize the initial loan. We first study optimal standard debt contracts. We deem a contract to be a standard debt contract if it has the following design. First, the household promises to meet a deterministic, possibly time-dependent, repayment schedule. Second, the household gets evicted if it fails to meet a payment.

The contracts with fixed repayments considered so far fall into this category. In general, however, a standard debt contract is characterized by its repayment schedule, which is a function g of the time passed since origination. More precisely, if the contract is originated at date t, then the $t + s$ contractual repayment is $e^{-g(s)}$. We have the following compact characterization of an optimal standard debt contract. For simplicity, we focus again on the case where $\chi = 0$.

PROPOSITION 2: *Let*

$$T_g = \min \left\{ \tau \geq 0 \ : \ W_{j,t+\tau} - W_{j,t} = \frac{\sigma \tau}{2} - \frac{g(\tau)}{\sigma} \right\},$$

and

$$Q_g(\tau) = Prob(T_g \geq \tau).$$

The optimal contract g solves

$$\sup_g \frac{\int_0^\infty Q_g(s) e^{-g(s)-(r+\delta)s} ds}{\lambda + \rho r \int_0^\infty Q_g(s) e^{-(r+\delta)s} ds}. \tag{8}$$

Proof: See the Appendix.∎

Unfortunately, problem (8) does not seem to be analytically solvable. Solving it numerically is also challenging.[11] We take first steps in this direction by studying the equilibrium when the contract space is expanded to contracts with repayments that

are piecewise log-linear. Specifically, we consider a space of contracts such that

$$g(s) = a + bs,$$

where constants a and b can take up to two different values on two intervals partitioning $[0, +\infty)$. We derive analytical expressions for the loan-to-income ratio, average income, and the default intensities given such a contract. Solving for the equilibrium contract is then a standard low-dimensional numerical search for extrema. Figure 3, Panel (A) compares optimal repayment schedules for the cases in which repayments are constant, log-linear, and piecewise log-linear.

[Figure 3 About Here]

Consider first the piecewise log-linear contract. It consists of high initial repayments that decrease rapidly followed by increasing repayments. It is interesting to notice that repayment ratios become optimally larger than one in the long run even though the income process has no drift and there is no inflation risk in our environment. These repayment patterns are reminiscent of teaser rates after an initial downpayment.

The intuition behind such schedules is as follows. At the beginning of the contract, banks perfectly observe borrowers' income and thus optimally extract a high fraction of it. As time elapses, two effects compete with each other. On the one hand, income distribution at a given date becomes more volatile, which induces banks to lower repayments in order to mitigate default risk. On the other hand, households' expected income at remote dates conditional on still honoring repayments increases over time, which calls for larger repayments. With a lognormal distribution, the dominant effect is initially the former, and then the latter.

If the contract space is restricted to log-linear repayments, then the slope of the optimal schedule depends on the expected length of the contract. For δ sufficiently small, banks expect a long-lasting contract, and therefore ask for increasing repayments in an effort to maximize the loan size. This case is depicted in Figure 3. Conversely, optimal repayment schedules are decreasing for larger values of δ. In this case, banks expect contracts to be short-lived and thus optimally focus on extracting high initial repayments.

Figure 3, Panel (B) illustrates equilibrium loan-to-income ratios with these three types of contracts as a function of δ. Overall, all three types of contracts deliver comparable loan-to-income ratios. For δ around 6%, loan-to-income ratios increase by less than 1% when we extend the contract space to log-linear contracts, and less than 5% by allowing for piecewise log-linear contracts. This suggests that the restriction to

fixed-repayment contracts is not critical for our results. Figure 3, Panels (C) and (D) show how steady-state default intensity and home prices vary with δ for the three types of contracts. It can be seen that the impact of the contract space on aggregate debt capacity is rather limited for δ around 6% — the value that we set to fit the unconditional income distribution. On the other hand, default intensity increases more significantly when the contract space becomes richer. The broad intuition is that more flexibility in contract design induces banks to seek higher future repayments from lucky borrowers against the risk of higher remote defaults. With flat repayment schedules, eliciting higher repayments comes at the cost of more early defaults that are very costly, and that time-dependent schedules can reduce.

In general, banks can commit to contingent repayment schedules. That is, at each date t after the outset of the contract, the bank can condition the eviction decision on the entire history of income reports and repayments made by the household between the beginning of the contract and t. We formally define such a general contingent repayment schedule in the Internet Appendix.

In the limiting case in which households are completely myopic ($R = +\infty$), such contingent schedules do not improve upon standard debt contracts. We offer a formal proof in the Internet Appendix. The intuition is simple. A myopic household, which cannot afford a bigger home at a given date t, maximizes date-t utility simply by repaying as little as possible while avoiding eviction. Since households repay the minimal amount that shields them from eviction at each date, no new information accrues to the bank regarding the household's income as history unfolds, except for the fact that the household's income is above this minimal repayment. Thus, there is no point in revising the repayment schedule based on the contract history. The household's income distribution conditional on the public history between zero and t is identical to the household's income distribution viewed from date 0 conditional on the contract not having been terminated before date t.

With a finite R, contingent repayment schedules could in principle improve upon noncontingent ones. Piskorski and Tchistyi (20011) characterize such optimal schedules in the case in which the borrower's income realizations are i.i.d. In our setup in which, consistent with the data, households' income shocks are persistent, solving for such optimal contracts is much more challenging. Fernandes and Phelan (2000) and Doepke and Townsend (2001) offer numerical solutions to contracting problems with persistent private information. Recent papers by Farhi and Werning (2011) and Williams (2011) offer characterizations of optimal contracts under persistent private information using first-order conditions. However, to the best of our knowledge, no closed-form solutions exist in general.

Finally, it is worth stressing that one obvious advantage of noncontingent contracts is their robustness. All that the bank needs to know to design an optimal debt contract is that borrowers are sufficiently constrained and impatient. In contrast, optimal contingent contracts are usually sensitive to the exact formulation of households' preferences.

V. Systematic Income Risk

This section offers insights into the impact of aggregate income shocks on debt capacities and home prices. The main goal is to show that diversifiable and nondiversifiable income risks reduce equilibrium debt capacities through different channels in this economy. As a result, the degree of income comovement among households can have an ambiguous effect on borrowing capacities and home prices.

A. Equilibrium

Consider a modification of the baseline model in which the income process is identical across households. The common income process $(I_t)_{t \geq 0}$ obeys

$$\frac{dI_t}{I_t} = \sigma dW_t,$$

where W_t is a standard Brownian motion. As before, we assume that for each household $j \in [0, 1]$, there exists a Poisson process $(N_{j,t})_{t \geq 0}$ with intensity $\delta > 0$ such that at each arrival time (ET date), household j vacates its current home and re-enters the housing market for unmodelled reasons. Unlike in the baseline model, however, the income of a household, to preserve its systematic nature, is not reset at ET dates.

When households are ex post identical, trades in the housing market follow only two motives: the occurrence of an ET date and default. Absent cross-sectional heterogeneity, there is no other reason to enter the market since there is no possibility to move along the property ladder. Note also that, absent any additional contracting restrictions, such systematic income shocks would not cause default. Because the mortgage market is active at all dates, banks would perfectly filter income shocks out of the home price process, and thus could write contingent contracts. Households could in fact borrow I_t/r against their entire future income because ET dates merely amount to replacing a borrower with an identical one at no cost.

Since we are interested in the impact of aggregate shocks on debt capacity in the presence of noncontingent contracts, we assume in this section that it is not possible to

index contracts on aggregate risk. This is arguably realistic: mortgages may be indexed to short-term (nominal) rates in practice, but usually do not depend on aggregate variables such as a housing price index. As before we focus on the tractable debt contracts with fixed repayment schedules studied in Section II.. The next proposition characterizes the equilibrium in the presence of such contracts. Recall that

$$\alpha = -\frac{1}{2} + \sqrt{\frac{2(r+\delta)}{\sigma^2} + \frac{1}{4}}, \quad \rho = 1 - \lambda \left(1 + \frac{\delta}{r}\right).$$

PROPOSITION 3: *There is a unique equilibrium with fixed-repayment contracts that satisfies the transversality condition:*

$$\lim_{s \to +\infty} E_t \left(e^{-rs} P_{t+s}\right) = 0. \tag{9}$$

In this equilibrium, the repayment associated with a contract initiated at date t is $\kappa \times I_t$, where the repayment ratio κ is the unique solution within $[0,1]$ of

$$(\alpha + 1)\kappa^\alpha = 1 + \rho\alpha\kappa^{\alpha+1}. \tag{10}$$

The price of a home unit P_t is a linear function of aggregate income I_t,

$$P_t = \frac{1 - \kappa^\alpha}{1 - \rho\kappa^{\alpha+1}} \times \frac{\kappa I_t}{r} = \frac{\alpha}{\alpha+1} \times \frac{\kappa I_t}{r}, \tag{11}$$

where $\frac{\kappa\alpha}{\alpha+1} \times \frac{1}{r}$ is also the loan-to-income ratio offered in the loan market.

Proof. See the Appendix.∎

Equation (11) shows that when banks quote a repayment ratio κ, they discount promised repayments κI_t at a rate $r + \Delta r$, where the equilibrium spread Δr is equal to $\frac{r\kappa^\alpha(1-\rho\kappa)}{1-\kappa^\alpha}$. The spread Δr increases with respect to κ because default risk is increasing in κ. This increasing spread implies that the loan size as a function of κ has a hump shape, and is maximal for the equilibrium value of κ defined by (10). The rising probability of default more than offsets the increase in promised repayments when κ increases beyond this equilibrium value. This model may be viewed as a dynamic equilibrium version of the model of credit rationing developed by Williamson (1987). The spread corresponding to the equilibrium value of κ is equal to r/α.

It is instructive to consider a case of zero eviction costs. When $\lambda \to 0$, then ρ tends

to one. Therefore, using expressions (10) and (11), we can see that

$$\lim_{\lambda \to 0} \kappa = 1, \quad \lim_{\lambda \to 0} P_t = \frac{\alpha}{\alpha + 1} \times \frac{I_t}{r}.$$

As in the baseline model without aggregate uncertainty, the repayment ratio tends to one when eviction costs vanish. Unlike in the baseline model, however, the equilibrium loan-to-income ratio is still strictly smaller than $1/r$ in the limit. This reflects the fact that negative realizations of systematic income risk imply contemporaneous large supply of vacant units (due to foreclosures) and small demand (due to the low borrowing capacity of potential buyers).[12] Of course, this effect becomes marginal as $\delta \to +\infty$, in which case $\alpha \to +\infty$. In this case most trades are for exogenous reasons, and supply and demand in the home market become therefore independent in the limit.

It is worthwhile noting that the fact that banks do not internalize the impact of their choice of a repayment ratio κ on the value of collateral does not reduce their supply of funds. In other words, a social planner internalizing the impact of κ on home prices — that is, taking into account the fact that P depends on κ — cannot find a value of κ for which debt capacities or home prices are higher than in the competitive setting with price-taking banks.[13] This contrasts with models that mix collateral constraints and investment decisions, such as Bernanke and Gertler (1989) or Kiyotaki and Moore (1997). In these models, decentralized equilibria inefficiently reduce aggregate borrowing, because the negative balance sheet externalities that lenders create for each other lead to inefficient investment decisions. In our environment, in which asset supply is fixed and households always invest the entire amount of their loans in housing, agents do not create such negative externalities for each other through the channel of inefficient investment decisions.

B.   Costs and Benefits of Diversification

In this section, we compare loan-to-income ratios in the nondiversifiable and diversifiable income risk cases. We begin with a situation in which relocation costs are infinite. When $\chi = 0$, contracts are terminated in both cases for the same two reasons only — ET dates or default. Recall that in the case of nondiversifiable income risk, the loan-to-income ratio is

$$L = \frac{\kappa}{r} \times \frac{(1 - \kappa^\alpha)}{1 - \rho \kappa^{\alpha+1}}. \tag{12}$$

When $\chi = 0$ and income risk is diversifiable, (5) simplifies to

$$\lim_{\chi \to 0} L = \frac{\kappa}{r} \times \frac{1 - \kappa^\alpha}{1 - \rho \kappa^\alpha}. \tag{13}$$

Comparing (12) with (13) yields the following result:

PROPOSITION 4: *If $\chi = 0$, the equilibrium loan-to-income ratio when income risk is diversifiable is larger than that in the nondiversifiable case if and only if*

$$\rho \geq 0 \Longleftrightarrow \lambda \leq \frac{r}{r + \delta}. \tag{14}$$

Proof: Let κ_n and κ_d denote the respective equilibrium repayment ratios for the nondiversifiable and diversifiable income risk cases. If $\rho \geq 0$, then

$$\frac{\kappa_n \left(1 - \kappa_n^\alpha\right)}{1 - \rho \kappa_n^\alpha} \geq \frac{\kappa_d \left(1 - \kappa_d^\alpha\right)}{1 - \rho \kappa_d^\alpha} \geq \frac{\kappa_d \left(1 - \kappa_d^\alpha\right)}{1 - \rho \kappa_d^{\alpha+1}}.$$

If $\rho \leq 0$, then

$$\frac{\kappa_d \left(1 - \kappa_d^\alpha\right)}{1 - \rho \kappa_d^{\alpha+1}} \geq \frac{\kappa_n \left(1 - \kappa_n^\alpha\right)}{1 - \rho \kappa_n^{\alpha+1}} \geq \frac{\kappa_n \left(1 - \kappa_n^\alpha\right)}{1 - \rho \kappa_n^\alpha}. \blacksquare$$

The intuition for this result is as follows. The expected proceeds from a loan are the sum of three components: i) the promised repayments until default or an exogenous move, (ii) sale proceeds net of eviction costs in the case of default, and (iii) proceeds from selling the vacant home if the household moves for exogenous reasons. Whether income risk is diversifiable or not has no impact on the value of component (i). Diversifiable income risk implies that the income of a given household is uncorrelated with home market values at default and ET dates. This has a positive effect on component (ii), which represents situations in which individual income has done poorly, but a negative effect on component (iii) when individual income has done rather well. As δ and λ increase, a larger fraction of total loan value comes from component (iii) than from component (ii). Thus, diversification has an adverse impact in this case.

In the presence of a finite relocation cost, it is no longer clear that equilibrium loan-to-income ratios are larger with diversifiable income shocks even for values of δ and λ that satisfy (14). In this case, the costs of diversification are more important because the cross-sectional mobility of incomes implies that good borrowers exercise their option to climb up the property ladder more often. Table I reports the values of χ above which equilibrium debt capacity with diversifiable income risk is lower than that with nondiversifiable income risk for different values of δ.

The values of χ below which diversifiable income risk yields lower debt capacities become quickly very low as δ increases. Thus, the costs of diversification become quickly as important as the benefits from higher collateral values.

In sum, table I shows that in the presence of one-sided commitment, diversifiability of borrowers' income risk does not necessarily ease collateralized lending. The larger

Table I: Debt Capacity

δ	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.1
χ	0.97	0.88	0.83	0.76	0.71	0.68	0.65	0.62	0.59

The table presents values of χ above which equilibrium debt capacity with diversifiable income risk is lower than that with nondiversifiable income risk for different values of δ. Other parameters are as follows: interest rate $r = 2\%$, volatility of individual income $\sigma = 25\%$, and eviction costs $\lambda = 30\%$.

ex post heterogeneity of borrowers is a double-edged sword. On the one hand, the assets seized from unlucky borrowers can always be sold to luckier borrowers standing ready to snap them up. But the flip side of these higher liquidation proceeds is that the same lucky borrowers receive more outside options in equilibrium, the exercise of which imposes more costs on lenders. The possible benefits from diversification of borrowers' risks on collateral liquidity have been well identified in the literature on endogenous debt capacities (see, for example, Shleifer and Vishny (1992)). That the costs induced by ex post heterogeneity may more than offset these benefits when borrowers' outside options are an equilibrium outcome is, to our knowledge, a novel finding.

Finally, the reader may wonder whether our comparison of borrowing capacities is fair and the amount of total risk the same in both cases. Income is reset to one at ET dates in the diversifiable income risk case, but not in the nondiversifiable one. The answer is actually positive. The only reason we reset income in the diversifiable income risk case is to obtain a stationary cross-section of incomes for calibration purposes. Without resets, income distribution would no longer be stationary. However, the loan-to-income ratio L and the average income of households in the market I would be identical since the average income of all households following an ET date is equal to one, which is all that is used to compute L and I.

VI.   Concluding Remarks

We develop an analytically tractable model of secured lending to liquidity constrained borrowers. We derive equilibrium debt capacities, asset prices, and default and refinancing intensities. Our findings suggest orders of magnitude for aggregate quantities that should prevail in a stable subprime market. The model allows us to understand the effect of a number of structural changes. For instance, the model predicts that lower relocation costs lead to lower individual debt capacities and default risk, and yet to higher aggregate debt capacities and default rates.

Our model can be estimated, and its predictions can be tested, using household

level data. In particular, in our equilibrium, only a fraction of the population seeks to trade in the housing market at a given date. The income distribution of this active fraction is different from that of the overall population. We find this to be a distinctive and novel feature of our model. This implies that our setup imposes a tight equilibrium relationship between the aggregate characteristics of the subprime market (default intensities, trading volume, home prices) and the cross-section of actively trading subprime households compared with the unconditional one.

The impact of the degree of income comovement across households on equilibrium outcomes can also be tested. We predict that trading volume should be more stable and higher on average in areas where income shocks are less correlated across households. The composition of the trading volume also depends on income correlation. The proportion of distressed sales should be both more stable and smaller on average when income correlation is lower. Further, we predict that the reduction in the transaction cost parameter χ that occurred during the subprime boom must have led to a higher increase in transactions in areas with low income correlation. Assessing whether these predictions hold in the data could be an interesting avenue for future research.

We make a number of our simplifying assumptions for analytical tractability. These assumptions can be relaxed for the purpose of estimation. For example, we assume that households always exhaust their debt capacities, which results in a uniform repayment ratio for all households. We could instead indirectly specify the cross-section of household preferences by using a distribution of repayment ratios observed in the data, or any other distribution. Similarly, we could allow for heterogeneity in refinancing costs and in volatility of income shocks across households.

Finally, our approach to equilibrium secured lending to difficult borrowers could be applied to situations other than mortgage markets, for example, to small businesses and entrepreneurs.

Notes

[1] Contributions include Foote, Gerardi, Goette, and Willen (2008, 2009), Gorton (2008), Keys, Mukherjee, Seru, and Vig (2010), Mian and Sufi (2009, 2010), Piskorski, Seru, and Vig (2010), and Rajan, Seru, and Vig (2009).

[2] See, for example, Mian and Sufi (2009).

[3] See, for example, Alvarez and Jermann (2000), Kehoe and Levine (2001), Krueger, Lustig, and Perri (2008), and Hellwig and Lorenzoni (2009) for recent contributions.

[4] Following Becker and Mulligan (1997), one can interpret a high discount rate R as a lack of financial sophistication.

[5] An alternative interpretation is that the household exits the economy at such dates, and is replaced with a new one starting out with unit income.

[6] In our model, income shocks have a permanent effect. Gourinchas and Parker (2002) and Carroll and Samwick (1997) allow income shocks to have also a transitory component. They estimate the volatility of permanent and transitory components to be 15% and 60%, respectively.

[7] The Internet Appendix can be found in the online version of this article.

[8] The front-end ratio divides mortgage payments, real estate taxes, and home insurance premia by gross income. The back-end ratio adds other obligations such as credit card and automobile debt to the numerator.

[9] See, for example, "The Subprime Lending Crisis," Report and Recommendations by the Majority Staff of the Joint Economic Committee, U.S. Senate, October 2007.

[10] This contrasts with Mayer, Piskorski, and Tchistyi (2008), where lower prepayment penalties lead to higher interest rates and defaults in partial equilibrium because the loan size is fixed.

[11] A numerical solution involves computing the distribution of the first hitting time $Q_g(\cdot)$ for a smooth boundary $g(\cdot)$, which is a notoriously difficult problem (see, for example, Durbin and Williams (1992), and Wang and Pötzelberger (1997) for some advances).

[12]Thus, trading volumes goes down when home prices go up in our model, which is counterfactual. This is because there is no rationale for downpayment constraints in our environment. If we introduced the exogenous constraint that households can voluntarily move at an ET date only if they have built up a minimal amount of home equity at this date, then positive income realizations could entail a larger trading volume.

[13]To see this, note that solving for P as a function of κ in (A17), plugging the expression in (A16), and then maximizing over κ yields the same loan-to-income ratios and home prices as in the competitive case in which banks maximize over κ taking P constant.

REFERENCES

Alvarez, Fernando, and Urban Jermann, 2000, Efficiency, equilibrium, and asset pricing with risk of default, *Econometrica* 68, 775-797.

Becker, Gary, and Casey Mulligan, 1997, The endogenous determination of time preference, *Quarterly Journal of Economics* 112, 729-758.

Bernanke, Ben, and Mark Gertler, 1989, Agency costs, net worth, and business fluctuations, *American Economic Review* 79, 14-31.

Borodin, Andrei, and Paavo Salminen, 2002, Handbook of Brownian motion - *Facts and Formulae* (Birkhauser Verlag AG, Basel).

Campbell, John Y. , Stefano Giglio, and Parag Pathak, 2011, Forced sales and house prices, *American Economic Review* 101, 2108–2131.

Carroll, Christopher D., and Andrew A. Samwick, 1997, The nature of precautionary wealth, *Journal of Monetary Economics* 40, 41-71.

Doepke, Matthias, and Robert M. Townsend, 2001, Credit guarantees, moral hazard, and the optimality of public reserves, working paper, UCLA and University of Chicago.

Durbin, James, 1992, The first-passage density of the Brownian motion process to a curved boundary, *Journal of Applied Probability* 29, 291-304.

Dynan, Karen, Douglas Elmendorf, and Daniel Sichel, 2008, The evolution of household income volatility, *Brooking Papers on Economic Activity*.

Fabozzi, Frank, 2006, The handbook of mortgage-backed securities (McGraw-Hill, New York).

Farhi, Emmanuel, and Ivan Werning, 2012, Insurance and taxation over the life cycle, *Review of Economic Studies* (forthcoming).

Fernandes, Ana, and Christopher Phelan, 2000, A recursive formulation for repeated agency with history dependence, *Journal of Economic Theory* 91, 223-247.

Foote Christopher, Kristopher Gerardi, Lorenz Goette, and Paul Willen, 2008, Just the facts: an initial analysis of the subprime crisis", *Journal of Housing Economics* 17, 291-305.

Foote Christopher, Kristopher Gerardi, Lorenz Goette, and Paul Willen, 2009, Reducing foreclosures: no easy answers, *NBER Macroeconomics Annual*.

Gerardi Kristopher, Harvey Rosen, Paul Willen, 2010, The impact of deregulation and financial innovation on consumers: the case of the mortgage market, *Journal of Finance* 65, 333-360.

Gorton, Gary, 2009, The subprime panic, *European Financial Management* 15, 10-46.

Gourinchas, Pierre-Olivier, and Jonathan A. Parker, 2002, Consumption over the life cycle, *Econometrica* 70, 47-89.

Hellwig, Christian, and Guido Lorenzoni, 2009, Bubbles and self-enforcing debt, *Econometrica* 77, 1137-1164.

Kehoe, Timothy J., and David K. Levine, 2001, Liquidity constrained markets versus debt constrained markets, *Econometrica* 69, 575-598.

Keys, Ben, Tanmoy Mukherjee, Amit Seru, and Vikrant Vig, 2010, Did securitization lead to lax screening? Evidence from subprime loans, *Quarterly Journal of Economics* 125, 307-362.

Kiyotaki, Nobuhiro, and John Moore, 1997, Credit cycles, *Journal of Political Economy* 105, 211-248.

Krueger, Dirk, and Harald Uhlig, 2006, Competitive risk sharing contracts with one-sided commitment, *Journal of Monetary Economics* 53, 1661-1691.

Krueger, Dirk, Hanno Lustig, and Fabrizio Perri, 2008, Evaluating asset pricing models with limited commitment using household consumption data, *Journal of the European Economic Association* 6, 715-726.

Mayer, Chris, Tomasz Piskorski, and Alexei Tchistyi, 2012, The inefficiency of refinancing: why prepayment penalties are good for risky borrowers?, *Journal of Financial Economics* (forthcoming).

Mian, Atif, and Amir Sufi, 2009, The consequences of mortgage credit expansion: evidence from the U.S. mortgage default crisis, *Quarterly Journal of Economics* 124, 1449-1496.

Mian, Atif, and Amir Sufi, 2010, The Great Recession: lessons from microeconomic data, *American Economic Review Papers and Proceedings* 100, 1-10.

Mian, Atif, Kamelesh Rao, and Amir Sufi, 2012, Household balance sheets, consumption, and the economic slump, working paper, University of California Berkeley and University of Chicago.

Ortalo-Magné, Francois, and Sven Rady, 2006, Housing market dynamics: on the contribution of income shocks and credit constraints, *Review of Economic Studies* 73, 459-485.

Piskorski, Tomasz, and Alexei Tchistyi, 2011, Stochastic house appreciation and optimal mortgage lending, *Review of Financial Studies* 24, 1407-1446.

Piskorski, Tomasz, Amit Seru, and Vikrant Vig, 2010, Securitization and distressed loan renegotiation: evidence from the subprime mortgage crisis, *Journal of Financial Economics* 97, 369-397.

Rajan, Uday, Amit Seru, and Vikrant Vig, 2010, The failure of models that predict failure: distance, incentives and defaults, working paper, University of Chicago Booth

School of Business.

Shleifer, Andrei, and Robert Vishny, 1992, Liquidation values and debt capacity: a market equilibrium approach, *Journal of Finance* 47, 1343-1366.

Stein, Jeremy, 1995, Prices and trading volume in the housing market: a model with down-payment effects, *Quarterly Journal of Economics* 110, 379-406.

Wang, Liqun, and Klaus Pötzelberger, 1997, Boundary crossing probability for Brownian motion and general boundaries, *Journal of Applied Probability* 34, 54-65.

Williams, Noah, 2011, Persistent private information, *Econometrica* 79, 1233-1274.

Williamson, Stephen, 1987, Costly monitoring, loan contracts, and equilibrium credit rationing, *Quarterly Journal of Economics* 102, 135-145.

Appendix

We first state a number of auxiliary results that we use repeatedly throughout the proofs and that can be skipped at first reading.

AUXILIARY RESULT 1: *The density of T_κ is*

$$\varphi_\kappa(t) = -\frac{\ln \kappa}{\sigma\sqrt{2\pi t^3}} e^{\frac{-\left(-\frac{\sigma}{2}t - \frac{\ln \kappa}{\sigma}\right)^2}{2t}}.$$ (A1)

Its Laplace transform is

$$\mathcal{L}_\kappa(s) = e^{\frac{\ln \kappa}{\sigma}\left(-\frac{\sigma}{2} + \sqrt{2s + \frac{\sigma^2}{4}}\right)} = \kappa^{-\frac{1}{2} + \sqrt{\frac{2s}{\sigma^2} + \frac{1}{4}}}.$$ (A2)

Proof: See, for example, Borodin and Salminen (2002).∎

AUXILIARY RESULT 2: *Let X be a random variable independent from T_δ taking values in $[0, +\infty) \cup \{+\infty\}$ whose density has Laplace transform $\mathcal{L}(\cdot)$. Define $T_{min} = \min(X, T_\delta)$. Then the Laplace transform of the density of T_{min} is given by*

$$\frac{s\mathcal{L}(s + \delta) + \delta}{s + \delta}.$$

Proof: Straightforward computations.∎

AUXILIARY RESULT 3: *Let $B_t^\mu = \mu t + B_t$, where B_t is a standard Wiener process. Define the running minimum M_t^μ as*

$$M_t^\mu = \min_{0 \leq s \leq t} B_t^\mu.$$

For $t > 0$ and $x \geq y$, $y \leq 0$,

$$Prob[B_t^\mu \in dx, M_t^\mu \in dy] = \frac{2(x - 2y)}{\sqrt{2\pi t^3}} e^{\mu x - \mu^2 t/2 - \frac{(x-2y)^2}{2t}}.$$ (A3)

Proof: See Borodin and Salminen (2002), formula 1.2.8, p.252.∎

Proof of Lemma 1

We first show that if ψ and R are sufficiently large, each household takes the largest possible loan at the outset of a contract, and invests the entire amount in housing. The proof is in two steps. In step 1, we show that for any given loan size, the household

invests the entire loan in housing. In step 2, we show that the household takes the largest possible loan at the outset of a contract.

Step 1. Let κ, L, and P respectively denote the steady-state repayment ratio, loan-to-income ratio, and price of a housing unit. These endogenous variables depend on ψ and R but they are bounded above and away from zero for all R, and ψ bounded away from zero. Suppose that a household at time t signs a new contract, reports an income $\widehat{I}_t$, and thus obtains a loan of size $L\widehat{I}_t$. Having the funds, the household solves for the optimal date-t c_t and q_t. Because c_t is nondurable, it affects only current utility. Thus, it is sufficient to show that current utility is maximized when the household invests its loan entirely in housing. Maximizing current utility amounts to solving

$$\max_{c_t,q_t} u\left(c_t, \chi q_t\right)$$

s. t.

$$c_t + Pq_t \leq L\widehat{I}_t.$$

Thus, if $\psi > P/\chi$, the household invests its loan entirely in housing: $q_t = L\widehat{I}_t/P$.

Step 2. We now prove that for R sufficiently large, households take the largest possible loan at the outset of a contract, that is, $\widehat{I}_t = I_t$. Intuitively, for large R, the household wants to frontload its consumption as much as possible. For expositional simplicity, we provide a formal proof for the case of a linear utility function $u\left(c_t, q_t\right) = c_t + \psi q_t$. A proof for a general $u\left(c_t, q_t\right)$ follows a similar logic.

Let $V(I_t)$ be the expected utility of a household with income I_t. Because of the linearity assumption, $V(I_t) = VI_t$, where V is a constant. Let $x = \widehat{I}_t/I_t$ and let $\tau(x)$ be the stopping time of contract termination if the household reports $\widehat{I}_t$. Then V solves

$$V = \max_{x\in[0,1]} E_t \int_0^{\tau(x)} Re^{-Rs}\left(I_{t+s}/I_t + \varphi_s x\right) ds + VRe^{-R\tau(x)}I_{t+\tau(x)}/I_t, \qquad \text{(A4)}$$

where

$$\varphi_s = \begin{cases} \varphi^- \equiv \chi\psi L/P - \kappa & \text{if} \quad s \in [0, \Delta t], \\ \varphi^+ \equiv \psi L/P - \kappa & \text{if} \quad s > \Delta t. \end{cases}$$

Suppose that $\psi > \kappa P/(\chi L)$ so that $\varphi^+ > \varphi^- > 0$. We need to show that for R large enough, the right-hand side of (A4) is maximized at $x = 1$. Let D be the derivative of

30

the right-hand side of (A4) with respect to x. Direct computations show that

$$D \ge \varphi^- + \frac{d}{dx}E_t \left(\underbrace{-e^{-R\tau(x)}\varphi^+ x}_{D_1} + \underbrace{\int_0^{\tau(x)} Re^{-Rs}\left(I_{t+s}/I_t\right)ds}_{D_2} + \underbrace{VRe^{-R\tau(x)}I_{t+\tau(x)}/I_t}_{D_3} \right).$$

(A5)

Consider term D_1. Let $f(t, x)$ be the density of $\tau(x)$. Thus,

$$E_t e^{-R\tau(x)} = \int_0^\infty e^{-Rt} f(t, x)dt.$$

It is well known that $f(t, x)$ is a bounded and smooth function of t and x (see, for example, Borodin and Salminen (2002)). Therefore,

$$D_1 = -\frac{d}{dx}\left(E_t e^{-R\tau(x)}\varphi^+ x\right) = -\varphi^+ \left(\int_0^\infty e^{-Rt} f(t, x)dt + x \int_0^\infty e^{-Rt} f'_x(t, x)dt \right). \quad \text{(A6)}$$

From (A6) it is clear that D_1 goes to zero as $R \to \infty$. Similarly, one can show that both D_2 and D_3 also go to zero as $R \to \infty$. Thus, D is positive for all $x \in [0, 1]$ for R large enough, and therefore the optimal choice of $\widehat{I}_t = I_t$.

Finally, we show that when $\psi, R \to \infty$, the upper termination threshold of a contract initiated at date t tends to $\chi^{-1} I_t$. When R becomes arbitrarily large, the moving decision is driven by the current utility. In this limiting case, the upper termination threshold I^* solves

$$u\left(I^* - \kappa I_t, \frac{LI_t}{P} \right) = u\left(I^*\left(1 - \kappa\right), \chi\frac{LI_t^*}{P} \right).$$

Clearly, $I^* \to \chi^{-1} I_t$ when ψ becomes arbitrarily large.

Proof of Formulae in Proposition 1

We use the following mathematical results, which can be found in Borodin and Salminen (2002, p. 627). Let $\omega(s) = \sqrt{\frac{2s}{\sigma^2} + \frac{1}{4}}$. For all $s > 0$,

$$E\left(e^{-sT_\chi}\mathbf{1}_{\{T_\chi<T_\kappa\}}\right) = \frac{\sqrt{\chi}\left(\kappa^{\omega(s)} - \kappa^{-\omega(s)}\right)}{(\chi\kappa)^{\omega(s)} - (\chi\kappa)^{-\omega(s)}}, \quad \text{(A7)}$$

$$E\left(e^{-sT_\kappa}\mathbf{1}_{\{T_\kappa<T_\chi\}}\right) = \frac{\chi^{\omega(s)} - \chi^{-\omega(s)}}{\sqrt{\kappa}\left((\chi\kappa)^{\omega(s)} - (\chi\kappa)^{-\omega(s)}\right)}. \quad \text{(A8)}$$

Formulae (3) and (4) follow from (A7) and (A8) by setting $s = \delta$.

Computation of $E\left(e^{-rT_\kappa}\mathbf{1}_{\{T_\kappa<T_\chi,T_\kappa<T_\delta\}}\right)$ **and** $1-Ee^{-rT}$ **in (7):**

Let $\varphi_\kappa(\cdot)$ denote the density of T_κ. Note that

$$E\left(e^{-rT_\kappa}\mathbf{1}_{\{T_\kappa<T_\chi,T_\kappa<T_\delta\}}\right)=\int_0^\infty e^{-(\delta+r)t}E\left(\mathbf{1}_{\{T_\chi>t\}}\right)\varphi_\kappa(t)dt=E\left(e^{-(r+\delta)T_\kappa}\mathbf{1}_{\{T_\kappa<T_\chi\}}\right),\tag{A9}$$

which can be computed using (A8) and setting $s=r+\delta$. Next, using Auxiliary Result 2 we have that

$$1-Ee^{-rT}=\frac{r}{r+\delta}\left(1-\frac{\sqrt{\chi}\left(\kappa^{\omega(r+\delta)}-\kappa^{-\omega(r+\delta)}\right)}{(\chi\kappa)^{\omega(r+\delta)}-(\chi\kappa)^{-\omega(r+\delta)}}-\frac{\chi^{\omega(r+\delta)}-\chi^{-\omega(r+\delta)}}{\sqrt{\kappa}\left((\chi\kappa)^{\omega(r+\delta)}-(\chi\kappa)^{-\omega(r+\delta)}\right)}\right).\tag{A10}$$

Proof of Proposition 2

Clearing the market for vacant homes in the presence of such contracts yields:

$$L=E\left(\int_0^{\min(T_\delta,T_g)}e^{-rs-g(s)}ds\right)+(1-\lambda)\,L\times E_t\left(e^{-rT_g}\mathbf{1}_{\{T_\delta>T_g\}}\right)+L\times E_t\left(e^{-rT_\delta}\mathbf{1}_{\{T_\delta\leq T_g\}}\right).$$

We have

$$
\begin{aligned}
E\left(\int_0^{\min(T_\delta,T_g)}e^{-rs-g(s)}ds\right) &= \int_0^\infty -\frac{d}{dt}\left(e^{-\delta t}Q_g(t)\right)\int_0^t e^{-rs-g(s)}ds\\
&= \int_0^\infty Q_g(s)\,e^{-(r+\delta)s-g(s)}ds,\\
E_t\left(e^{-rT_\delta}\mathbf{1}_{\{T_\delta\leq T_g\}}\right) &= \int_0^\infty \delta e^{-(r+\delta)t}Q_g(t)\,dt,
\end{aligned}
$$

$$E\left(e^{-rT_g}\mathbf{1}_{\{T_\delta>T_g\}}\right)=\int_0^\infty \frac{d}{dt}\left[1-Q_g(t)\right]e^{-(r+\delta)t}dt=(r+\delta)\int_0^\infty\left(1-Q_g(t)\right)e^{-(r+\delta)t}dt$$

Solving for L yields the proposition.∎

Proof of Proposition 3

We need to solve for the loan-to-income ratio and for the repayment ratio κ that a bank chooses to offer at date t under the expectation that future home prices will satisfy

$$\forall u\geq 0, P_{t+u}=PI_{t+u}\tag{A11}$$

for some constant $P>0$.

In the case of systematic risk, the income of any household perfectly correlates with the home price. Given the fixed-repayment contracts, borrowing capacities are also linear in income. Borrowing capacities and home prices therefore move in lockstep. As a result, the household cannot buy a larger home when its income increases. Thus, households never move voluntarily and the upper termination threshold is infinite.

With the notations introduced in the proof of Proposition 1, a loan taken at date t is terminated at the random date $t + T$, where

$$T = \min \left(T_\delta, T_\kappa \right).$$

If a household j accepts the offer from a bank that quotes a repayment ratio κ at date t, the bank expects the future flows from lending to household j to be equal to

$$LI_{j,t} = E_t \left(\int_t^{t+T} e^{-rs} \kappa I_{j,t} ds + (1 - \lambda) \, 1_{\{T_\delta > T_\kappa\}} e^{-rT_\kappa} P_{t+T_\kappa} + 1_{\{T_\delta \leq T_\kappa\}} e^{-rT_\delta} P_{t+T_\delta} \right). \tag{A12}$$

Applying (A10) with $\chi = 0$ we have

$$E_t \left(\int_t^{t+T} e^{-rs} ds \right) = E_t \left(\frac{1 - e^{-rT}}{r} \right) = \frac{1 - \kappa^\alpha}{r + \delta}. \tag{A13}$$

Using Auxiliary Result 1, we can see that

$$E_t \left(e^{-rT} 1_{\{T_\delta > T_\kappa\}} I_{t+T} \right) = \kappa I_t \int_0^\infty e^{-(r+\delta)\tau} \varphi_\kappa(\tau) d\tau = \kappa^{\alpha+1} I_t. \tag{A14}$$

Finally, we use the following result, which we prove a bit later:

$$E_t \left(e^{-rT} 1_{\{T_\delta \leq T_\kappa\}} I_{t+T} \right) = \frac{\delta \left(1 - \kappa^{\alpha+1} \right)}{r + \delta} I_t. \tag{A15}$$

Market clearing and (A13), (A14), and (A15) imply that

$$LI_t = PI_t = \left(\frac{\kappa \left(1 - \kappa^\alpha \right)}{r + \delta} + \left((1 - \lambda) \kappa^{\alpha+1} + \frac{\delta \left(1 - \kappa^{\alpha+1} \right)}{r + \delta} \right) P \right) I_t, \tag{A16}$$

or

$$L = P = \frac{\kappa \left(1 - \kappa^\alpha \right)}{r + \delta} + P \times \left(\frac{\delta}{r + \delta} + \frac{\rho r \kappa^{\alpha+1}}{r + \delta} \right). \tag{A17}$$

Solving for P we obtain (11). From (A17) it is clear that the equilibrium repayment ratio κ that maximizes L also maximizes P, and thus $\frac{\partial P}{\partial \kappa} = 0$ in the equilibrium. From

([A17](#)) it is easy to see that the optimal κ is a constant such that

$$(1 + \alpha)\,(1 - r\rho P)\,\kappa^\alpha = 1. \tag{A18}$$

Proposition [3](#) is then obtained by plugging ([11](#)) into ([A18](#)). That ([10](#)) has a unique solution over $[0, 1]$ is easy to see with a monotonicity argument.

Now we prove that

$$E_t\left(e^{-rT}1_{\{T_\delta \leq T_\kappa\}}I_{t+T}\right) = \frac{\delta\left(1 - \kappa^{\alpha+1}\right)}{r + \delta}I_t.$$

Let us introduce a family of stopping times $(T_{\kappa,\tau})_{\tau \geq 0} = (\min(T_\kappa, \tau))_{\tau \geq 0}$. We have

$$E_t\left(e^{-rT}\mathbf{1}_{\{T_\kappa \geq T_\delta\}}I_{t+T}\right) = \int_0^\infty \delta e^{-\delta\tau}e^{-r\tau}E_t[I_{t+\tau}\mathbf{1}_{\{T_{\kappa,\tau}=\tau\}}]d\tau.$$

Notice that

$$\{T_{\kappa,\tau} = \tau\} = \{\min_{0 \leq s \leq \tau} W_{t+s} - W_t - \frac{\sigma s}{2} > \frac{\ln(\kappa)}{\sigma}\} \quad a.s.$$

By ([A3](#)) we have

$$E_t\left(I_{t+\tau}\mathbf{1}_{\{T_{\kappa,\tau}=\tau\}}\right) = I_t \int_{\frac{\ln(\kappa)}{\sigma}}^0 \int_y^\infty \frac{2(x - 2y)}{\sqrt{2\pi\tau^3}}e^{-\frac{(x-2y)^2}{2\tau}}e^{\sigma x}e^{-\frac{\sigma x}{2} - \frac{\sigma^2\tau}{8}}dxdy$$

$$= I_t \int_{\frac{\ln(\kappa)}{\sigma}}^0 \int_0^\infty \frac{2(x - y)}{\sqrt{2\pi\tau^3}}e^{-\frac{(x-y)^2}{2\tau}}e^{\frac{\sigma}{2}(x+y)}e^{-\frac{\sigma^2\tau}{8}}dxdy.$$

Using ([A2](#)) we have

$$\int_0^\infty \delta e^{-(\delta+r+\frac{\sigma^2}{8})\tau}\frac{2(x - y)}{\sqrt{2\pi\tau^3}}e^{-\frac{(x-y)^2}{2\tau}}d\tau = 2\delta e^{-\sqrt{2(r+\delta)+\frac{\sigma^2}{4}}(x-y)}, \quad x \geq y.$$

By Fubini's theorem,

$$E_t\left(e^{-rT}\mathbf{1}_{\{T_\kappa \geq T_\delta\}}I_{t+T}\right) = \delta I_t \int_{\frac{\ln(\kappa)}{\sigma}}^0 \int_0^\infty e^{-\sqrt{2(r+\delta)+\frac{\sigma^2}{4}}(x-y)}e^{\frac{\sigma}{2}(y+x)}dxdy \tag{A19}$$

$$= \frac{\delta I_t}{r + \delta}\left(1 - \kappa^{\frac{1}{2}+\sqrt{\frac{2(r+\delta)}{\sigma^2}+\frac{1}{4}}}\right).\blacksquare \tag{A20}$$
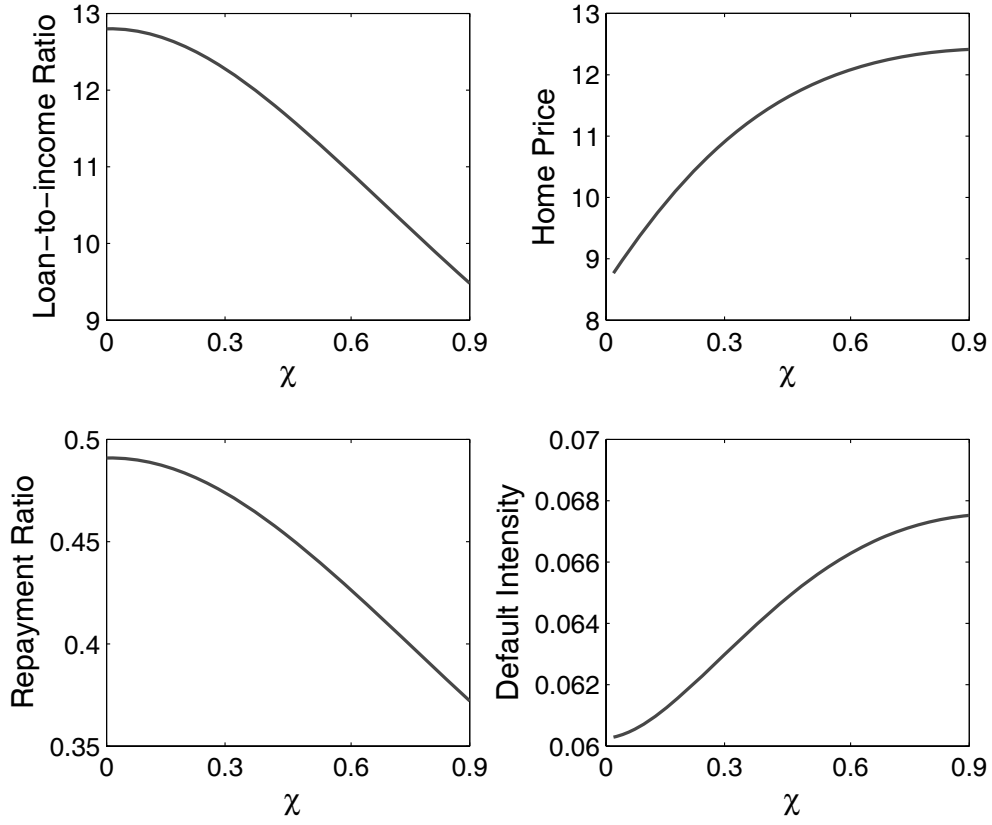
Figure 1. Comparative statics. Figure 1 shows the equilibrium values of the loan-to-income ratio L, the home price P, the repayment ratio κ, and the default rate under an optimal contract with fixed repayments as a function of the relocation cost parameter χ (its transformation $\chi^{-1} - 1$ defines the relocation costs). Other parameters are as follows: interest rate $r = 2\%$, volatility of individual income $\sigma = 25\%$, and eviction costs $\lambda = 30\%$.
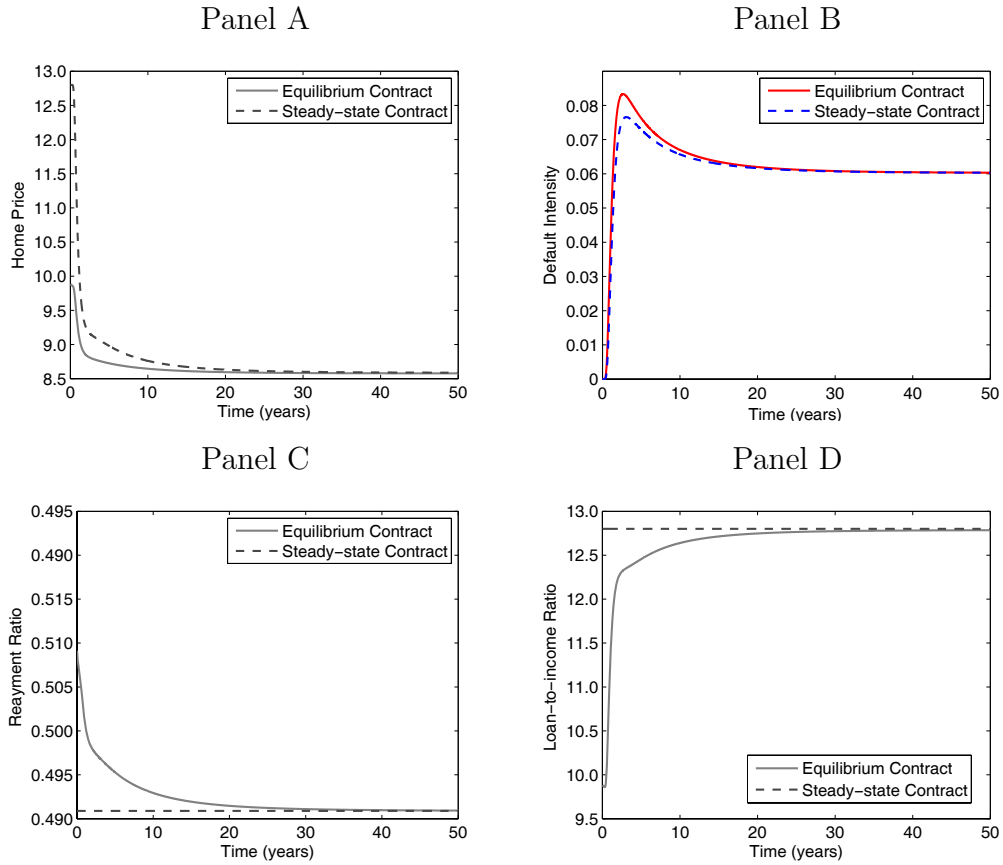
Figure 2. Transition dynamics. Figure 2 shows the evolution of aggregate quantities along the transition path to the steady-state described in Section III.. Panel A displays the evolution of the home price. Panel B displays the evolution of default intensities. Panel C displays the evolution of the repayment ratio κ. Panel D displays the loan-to-income ratio L. The dashed line shows the evolution of aggregate quantities if banks were naively offering the steady-state contract from date 0. The solid line shows the evolution that prevails in equilibrium when banks optimize fixed-repayment contracts at each date t. Other parameters are as follows: interest rate $r = 2\%$, volatility of individual income $\sigma = 25\%$, eviction costs $\lambda = 30\%$, and relocation cost parameter $\chi = 0$.
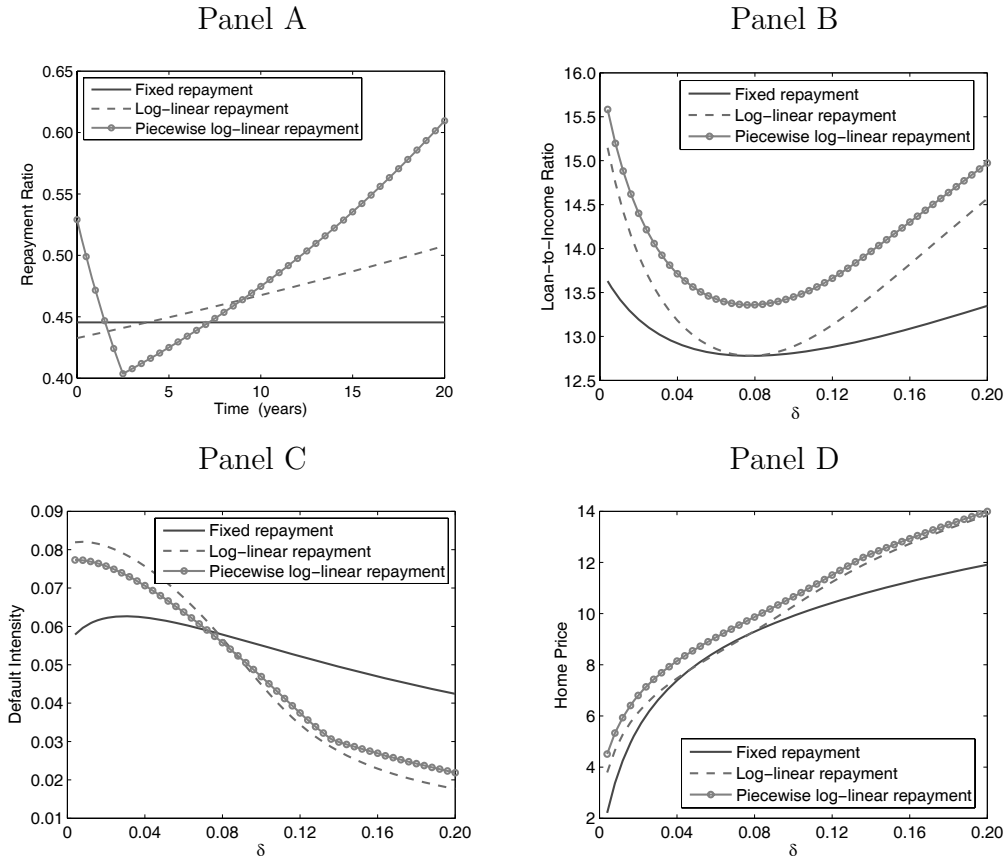
Figure 3. Optimal contracts. Figure 3, Panel A displays the repayment ratio κ_t as a function of time passed since the outset of the contract for the optimal contracts described in Section IV in three different contract spaces. Figure 3, Panels B to D show the behavior of aggregate quantities in the three contract spaces described in Section IV. for different values of the arrival intensity of ET dates δ. Panel B displays the loan-to-income ratio. Panel C displays the default intensity. Panel D displays the home price. Other parameters are as follows: interest rate $r = 2\%$, volatility of individual income $\sigma = 25\%$, eviction costs $\lambda = 30\%$, and relocation cost parameter $\chi = 0$.

Internet Appendix for
"Equilibrium Subprime Lending"[*]

This Internet appendix details the methodologies that we use in three parts of the paper. In section A, we derive the cross-sectional distribution of household income and explain how we estimate it using the PSID data set. This relates to the section II. B of the paper. In section B, we explain how we solve for the transition dynamics that are the scope of the section III of the paper. Finally, in section C, we prove the claim made in the section IV of the paper that the optimal contract is a standard debt contract when households are myopic.

A. Income Distribution

Since the income of each household follows a geometric Brownian motion and is reset to one at Poisson random times (ET dates), the cross-section of household income has the distribution of the exponentially stopped geometric Brownian motion:

$$P(I \in dz) = \begin{cases} \frac{\delta}{\omega \sigma^2} z^{-\frac{3}{2}-\omega} & \text{if} \quad z \geq 1 \\ \frac{\delta}{\omega \sigma^2} z^{-\frac{3}{2}+\omega} & \text{if} \quad z \leq 1 \end{cases}, \quad \omega = \sqrt{\frac{2\delta}{\sigma^2} + \frac{1}{4}}. \tag{IA1}$$

Proof: Formula 1.0.5 in Borodin and Salminen (2002, p. 606). The distribution (IA1) depends on the parameter ω. The log-likelihood function is

$$N \ln \left(\frac{\omega^2 - \frac{1}{4}}{2\omega} \right) + \omega \left(\sum_{z_i < 1} \ln z_i - \sum_{z_i > 1} \ln z_i \right). \tag{IA2}$$

Therefore, ω is a solution to the following equation

$$\frac{\omega^2 + \frac{1}{4}}{\omega(\omega^2 - \frac{1}{4})} = \frac{\sum_{z_i > 1} \ln z_i - \sum_{z_i < 1} \ln z_i}{N}. \tag{IA3}$$

Using household income from PSID data for 2003, we find that the right-hand side of (IA3) is 0.8174. Solving for ω gives $\omega = 1.52$. Our estimate of $\sigma = 0.25$ then implies that $\delta = 0.064$. As an attempt to better match the subprime population, we also perform the same estimation for the following subsets of the population: PSID race code 2 (African-American), 12 or less years of schooling, and age below 50. The

respective values of ω for each subset are 1.42, 1.57, and 1.51, which correspond to values of $\delta = 0.055\%$, $\delta = 0.069\%$, and $\delta = 0.063\%$. In our calibration exercise, therefore, we set $\delta = 0.0625$, which corresponds to $\omega = 1.5$.

B. Transition Dynamics

B.1. Home Prices P_t and Average Home Supply S_t

We first show that for all $t > 0$, $S_t = S_0 = 1$ and $P_t = L_t I_t$. Market clearing implies that for all t,

$$P_t S_t = L_t I_t.$$

Thus, we only need to show that $S_t = 1$ for all $t > 0$. For $t > 0$, $\tau \in [0, t)$, let $\pi_{\tau, t}$ be the fraction of households in the market at date t whose previous market participation took place at date τ. Notice that the probability of being in the market at date t, given that the previous participation occurred at date τ, does not depend on income at date τ. This is because both ET and default dates on a given loan are independent from the income level at loan origination. The average income of the above fraction at time τ therefore equals I_τ, and hence the average supply of housing units of the above fraction is S_τ. Thus, S_t solves

$$S_t = \int_0^t \pi_{\tau, t} S_\tau d\tau, \quad S_0 = 1, \quad t \in (0, \infty). \tag{IA4}$$

Equation (IA4) is a Volterra integral equation of the second kind. Notice that,

$$\pi_{\tau, t} d\tau = dF(t - \tau)/F(t), \tag{IA5}$$

where $F(\cdot)$ is the c.d.f. of the stopping time T that describes the realized duration of a loan. Define an operator $A^t : C[0, t] \to C[0, t]$ as

$$A^t f(s) = \int_0^s \pi_{\tau, s} f(\tau) d\tau. \tag{IA6}$$

Since

$$\forall t \quad \int_0^t \pi_{\tau, t} d\tau = 1,$$

the operator A^t has a unit norm. To complete the proof, it suffices to show that there exists no eigenfunction that takes zero value at zero and has eigenvalue one. Suppose conversely that there exists such a function $f(s)$. Let $t^{max} = arg\max_{s \in [0, t]} |f(s)|$. Since

2

$f(0) = 0$ and $f \neq 0$, it must be that $t^{max} > 0$. However,

$$Af(t^{max}) = \int_0^{t^{max}} \pi_{\tau, t^{max}} f(\tau)d\tau < f(t^{max}),$$

a contradiction.

B.2. Calculation of Arrival Intensities and Average Income

Let T^i be a stopping time that measures the time passed until household i, currently in the market, returns to the market. Since $\chi = 0$, the household returns to the market only because of default or at ET dates. Since household incomes are independent, T^i are also mutually independent across households. In what follows, therefore, we drop the superscript i.

First consider the case in which banks offer the steady-state fixed-repayment contract (κ, L) at all dates t. Recall that in this case

$$T = \min\{T_\kappa, T_\delta\}.$$

Let φ denote the density of T. It is well known that the arrival intensity of households to the market between t and $t + dt$, $\varphi^*(t)$, is the solution to the renewal equation

$$\varphi^*(t) = \varphi(t) + \int_0^t \varphi^*(t - s)\varphi(s)ds. \tag{IA7}$$

Define ϕ_κ and ϕ_δ as

$$\phi_\kappa(t)dt \equiv P\{T \in dt, T = T_\kappa\} = e^{-\delta t} \frac{\ln \kappa}{\sigma \sqrt{2\pi t^3}} e^{-\left(\frac{\sigma}{2}t + \frac{\ln \kappa}{\sigma}\right)^2 / 2t}, \tag{IA8}$$

$$\phi_\delta(t)dt \equiv P\{T \in dt, T = T_\delta\} = \delta e^{-\delta t} \left(1 - F_\kappa(t)\right) dt, \tag{IA9}$$

$$F_\kappa(t) = \frac{1}{2} \left(\text{Erfc} \left[\frac{1}{\sqrt{2t}} \left(a - \frac{\sigma t}{2} \right) \right] + \kappa^{-1} \text{Erfc} \left[\frac{1}{\sqrt{2t}} \left(a + \frac{\sigma t}{2} \right) \right] \right). \tag{IA10}$$

The default intensity $\varphi_\kappa^*(t)$ at time t solves

$$\varphi_\kappa^*(t) = \phi_\kappa(t) + \int_0^t \varphi^*(t - s)\phi_\kappa(s)ds. \tag{IA11}$$

The average income I_t of households in the market at time t solves

$$I_t \varphi^*(t) = \kappa \phi_\kappa(t) + \phi_\delta(t) + \int_0^t \varphi^*(t - s) \left(\kappa I_{t-s} \phi_\kappa(s) + \phi_\delta(s) \right) ds. \tag{IA12}$$

3

Since the ET dates arrive with Poisson intensity, formula (IA12) can also be written as

$$I_t \phi^*(t) = \kappa \phi_\kappa(t) + \delta + \kappa \int_0^t \phi^*(t-s) I_{t-s} \phi_\kappa(s) ds. \tag{IA13}$$

Similarly, $\varphi^*(t)$ solves

$$\varphi^*(t) = \phi_\kappa(t) + \delta + \int_0^t \varphi^*(t-s) \phi_\kappa(s) ds. \tag{IA14}$$

B.3. Computations with Equilibrium Contracts

Step 1 First, we start with the steady-state contract characterized by (κ, L). For this contract, we compute $\phi_\kappa(t)$, $\phi_\delta(t)$, the arrival intensity $\phi^*(t)$, and the average income I_t using formulas (IA8), (IA9), (IA7), and (IA12).

Step 2 Given the income process I_t, recompute κ_t and L_t to ensure that banks maximize the loan size and break even.

Suppose that a household has income of one. Then it can afford a home of size

$$\frac{L_t}{P_t} = \frac{1}{I_t}$$

Thus, a bank expects to recover

$$\frac{\kappa_t \left(1 - \kappa_t^\alpha\right)}{r + \delta} + E_t \left[e^{-rT} \left(1 - \lambda \mathbf{1}_{\{T_\kappa < T_\delta\}}\right) L_{t+T} I_{t+T} / I_t \right]. \tag{IA15}$$

Therefore,

$$L_t = \max_{\kappa_t} \frac{\kappa_t \left(1 - \kappa_t^\alpha\right)}{r + \delta} + E_t \left[e^{-rT} \left(1 - \lambda \mathbf{1}_{\{T_\kappa < T_\delta\}}\right) L_{t+T} I_{t+T} / I_t \right], \tag{IA16}$$

and

$$\kappa_t = Arg \max_{\kappa_t} \frac{\kappa_t \left(1 - \kappa_t^\alpha\right)}{r + \delta} + E_t \left[e^{-rT} \left(1 - \lambda \mathbf{1}_{\{T_\kappa < T_\delta\}}\right) L_{t+T} I_{t+T} / I_t \right]. \tag{IA17}$$

To compute (IA16) and (IA17), we use the fact that in the long-run $L_t = L$ and $\kappa_t = \kappa$. Thus, L_t and κ_t can be solved recursively going backwards.

Step 3 We recompute the arrival intensities $\phi^*(t)$, $\phi_\kappa(t)$, and $\phi_\delta(t)$ and average income I_t for the κ_t process using the following generalization of formulas (IA14), (IA11),

and (IA13):

$$\phi^*(t) = \phi(\kappa_0, t) + \delta + \int_0^t \phi^*(t-s)\phi_\kappa(\kappa_{t-s}, s)ds, \qquad \text{(IA18)}$$

$$\phi^*_\kappa(t) = \phi_\kappa(\kappa_0, t) + \int_0^t \phi^*(t-s)\phi_\kappa(\kappa_{t-s}, s)ds, \qquad \text{(IA19)}$$

$$I_t\phi^*(t) = \kappa_0\phi_\kappa(\kappa_0, t) + \delta + \int_0^t \phi^*(t-s)\kappa_{t-s}I_{t-s}\phi_\kappa(\kappa_{t-s}, s)ds. \qquad \text{(IA20)}$$

Step 4 Repeat steps 2 and 3 until convergence.

C. Contingent Repayment Schedules

For expositional simplicity, we adopt a discrete-time framework with dates $\{n\Delta t\}_{n\geq 0}$, where $\Delta t > 0$. This way, the game between banks and households is simpler to formalize. As in the continuous-time model, at each date n, ET dates are realized, and household j privately observes and consumes its income $I_{j,n\Delta t}$. The income process is a series of discrete snapshots from its continuous-time counterpart.

Consider a household that has taken a loan at date 0. A generic contingent repayment schedule can be described as follows. Let h_n denote public history at date n. That is, h_n is the collection of all public information and all observable actions taken by each party between zero and the end of period $[(n-1)\Delta t, n\Delta t]$, with $h_0 = \varnothing$. At the beginning of each period $[n\Delta t, (n+1)\Delta t]$, the household terminates the contract if it can afford a larger home or an ET date realizes, and stays in the contract otherwise. In the latter case, the household makes an income report $\widehat{I}_{j,n\Delta t} > 0$. The bank then asks for a repayment $\rho(h_n, \widehat{I}_{j,n\Delta t}) > 0$, and commits to evict the household if it fails to meet the repayment. As is common in applications of costly state verification models (see, for example, Bernanke and Gertler (1989) and Gale and Hellwig (1985)), we require eviction decisions to be a deterministic function of history.

It is easy to see that the optimal repayment schedule is noncontingent when the household is myopic and cares only for current utility. Suppose that at a given date $n \geq 1$, a household cannot afford a larger home and an ET date has not been realized. In this case, maximizing its current utility implies making the report $\widehat{I}_{j,n\Delta t}$ that minimizes the current repayment $\rho(h_n, \widehat{I}_{j,n\Delta t})$. By induction, the household will make a minimal repayment at each date given a history of having made minimal repayments before. Thus, any contingent contract generates the same repayments as the noncontingent contract comprised of deterministic repayments defined recursively as follows. The

5

first noncontingent repayment is the minimal repayment recommended at date 1 in the contingent contract over all possible date-1 reports. Then the date $n+1$ noncontingent repayment is the minimal date $n+1$ repayment of the contingent contract conditional on always having made such a minimal repayment before.

REFERENCES

Bernanke, Ben, and Mark Gertler, 1989, Agency costs, net worth, and business fluctuations, *American Economic Review* 79, 14-31.

Borodin, Andrei, and Paavo Salminen, 2002, Handbook of Brownian motion - *Facts and Formulae* (Birkhauser Verlag AG, Basel).

Gale, Douglas, and Martin Hellwig, 1985, Incentive-compatible debt contracts: the one-period problem, *Review of Economic Studies* 52, 647-663.